

On the geometry of the Batalin-Vilkovisky Laplacian

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Abstract. We define a jet-space analog of the BV-Laplacian, avoiding delta-functions and infinite constants; instead we show that the main properties of the BV-Laplacian and its relation to the Schouten bracket originate from the underlying jet-space geometry.

Key words: jet space, BV-formalism, Schouten bracket, BV-Laplacian, regularization, quantum master equation

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Introduction

The Batalin–Vilkovisky (BV) Laplacian Δ_{BV} is a necessary ingredient in the quantization of gauge-invariant systems of Euler–Lagrange equations [2, 3, 9]. Introduced after the generalization of the standard calculus on a smooth manifold (which can be viewed as the set of locations for a material point [1]) to the variational calculus of fields over space-time [7, 14, 19], the BV-Laplacian patterns upon the familiar construction $\Delta = d_{\text{dR}} * d_{\text{dR}}$ (here d_{dR} is the de Rham differential on a manifold and $*$ is the Hodge star). In the new geometry of gauge fields, the construction of the BV-Laplacian Δ_{BV} relies on the presence of canonically conjugate pairs of variables such as fields and antifields or (possibly, higher) ghost-antighost pairs, which stem from the derivation of the Euler–Lagrange equations of motion $\delta S = 0$ and (possibly, higher generations of) Noether relations, respectively (see [2, 3, 4, 5, 20] and [12, 14]). This relates the BV-Laplacian Δ_{BV} to the variational Schouten bracket \llbracket, \rrbracket (the odd Poisson bracket, or *antibracket* [22]), which measures for the operator Δ_{BV} its deviation from being a derivation.

The Laplace equation $\Delta_{\text{BV}}(F) = 0$ constrains the (products of) integral functionals F of fields and antifields in such a way that Feynman’s path integral of a Laplace equation’s solution F over the space of admissible fields is essentially independent of the non-physical antifields. As soon as the setup becomes quantum and all objects depend also on the Planck constant \hbar , the Laplace equation $\Delta(\mathcal{O}^{\hbar} \cdot \exp(iS_{\text{BV}}^{\hbar}/\hbar)) = 0$ selects the observables \mathcal{O}^{\hbar} (here $i^2 = -1$ and S_{BV}^{\hbar} is the extension in powers of \hbar for the full BV-action $S_{\text{BV}} = S + \dots$ of a given gauge-invariant model $\delta S = 0$). This approach yields the quantum master equation upon S_{BV}^{\hbar} and creates the cohomology groups with respect to the quantum BV-differential $\Omega = -i\hbar\Delta + \llbracket S_{\text{BV}}^{\hbar}, \cdot \rrbracket$. The observables are Ω -closed, $\Omega(\mathcal{O}^{\hbar}) = 0$; Feynman’s path integral is then used to calculate their expectation values and correlations.

On top of the difficulties which are immanent to a definition of the path integral, the BV-Laplacian Δ_{BV} itself often suffers from a necessity to be regularized manually if one wishes to avoid the otherwise appearing “infinite constants” or Dirac’s delta-distributions (e.g., see [6], in which such constants appear). We now argue that such difficulties, leading to a necessity to regularize the object Δ_{BV} , are brought into the picture by an incomplete utilization of the geometry at hand – so that the manual regularization procedure is in fact redundant: we claim that no sources of divergence are built into the genuine definition of Δ_{BV} . The intrinsic self-regularization of the BV-Laplacian, the necessity of which was long stated in the mathematical physics literature, is achieved in this paper.

We approach the regularization problem for the BV-Laplacian Δ_{BV} in terms of the geometry

which properly takes into account (i) the Euler–Lagrange equations of motion $\mathcal{E} = \{\delta S = \mathcal{E}_\alpha \delta \phi^\alpha = 0\}$ upon the physical fields ϕ^α , (ii) Noether’s identities $\Phi = \{\Phi^\mu(x, [\phi^\alpha], [\mathcal{E}_\beta]) \equiv 0\}$ that hold regardless of whether the fields satisfy the equations of motions, and possibly (iii) higher generations of constraints $\chi_i = \{\chi_i^b(x, [\phi^\alpha], [\mathcal{E}_\beta], [\Phi^\mu], \dots, [\chi_{i-1}^\nu]) \equiv 0\}$ between the equations \mathcal{E} , Noether’s identities Φ , etc. (see [14, Ch. 2]). The BV-setup is then achieved in several standard steps [2, 3]. We first recall that differential equations of any nature (e.g., $\mathcal{E}_\alpha = 0$ or $\Phi^\mu = 0$) map their arguments to spaces of sections of auxiliary bundles so that the pre-image of the zero section, which stands in the equations’ right-hand sides, presents particular interest; in section 1 below we recall in detail a construction of the modules $P_0 \ni \mathcal{E}$, $P_1 \ni \Phi$, $P_2 \ni \chi_2, \dots, P_\lambda \ni \chi_\lambda$ of equations, identities, and higher generations of constraints. For every $i = 0, \dots, \lambda$ we then introduce the dual space \widehat{P}_i of linear, form-valued functions on P_i by recognizing each \widehat{P}_i as the module of sections of the dual bundle induced over the product over $M \ni x$ of the infinite jet space $J^\infty(\pi)$ of physical fields ϕ^α and, whenever appropriate, jet spaces that encode Noether’s identities Φ , $\chi_2, \dots, \chi_{i-1}$. Thirdly, by using the concept of *neighbours* (c.f. [21]) we reverse the (ghost-)parity of the duals and obtain the odd-parity modules $\Pi \widehat{P}_i$.

In agreement with the BV-procedure, we now change the scene by ‘forgetting’ that differential constraints Φ , $\chi_2, \dots, \chi_\lambda$ involve the fields ϕ^α , equations of motion, and lower-order generations of identities.¹ Namely, we replace the respective objects of opposite ghost-parities by introducing the full set of BV-variables $(\mathbf{q}, \mathbf{q}^\dagger)$ and constructing the \mathbb{Z}_2 -graded bundle π_{BV} over the space-time M ; we retain the physical fields ϕ^α among q^a ’s but also produce² the odd *antifields* ϕ_α^\dagger as the respective part of \mathbf{q}^\dagger , and also the odd *ghosts* γ_μ and even-parity *antighosts* $\gamma^{\dagger, \mu}$ or higher ghost-antighost pairs $c^\alpha \leftrightarrow c_\alpha^\dagger$ (see also [10] in this context). This argument yields the BV-zoo of canonically conjugate pairs $q^a \leftrightarrow q_a^\dagger$ of (anti)fields or (anti)ghosts and their derivatives along M , that is, along the base manifold in the bundle π of physical fields.

In the course of the scene change, the form-valued coupling between the modules P_i of differential equations of all sorts and the neighbours $\Pi \widehat{P}_i$ takes in the BV-setup the shape of an \mathbb{R} - or \mathbb{C} -valued coupling \langle, \rangle between the variations of canonically conjugate variables: $\langle \delta q^a, \delta q_b^\dagger \rangle = +1 \cdot \delta_b^a = -\langle \delta q_b^\dagger, \delta q^a \rangle$ (c.f. [1, §37]). Given this coupling and a proper count of the ghost \mathbb{Z}_2 -parity (as given by the operator Π) and \mathbb{Z} -grading (stemming from differential forms such as the variations), the introduction of an odd Poisson bracket (3) is immediate. We proceed with a definition of the BV-Laplacian Δ_{BV} by Eq. (5); in a purely even initial setup, this yields a familiar coordinate expression for this operator:

$$\Delta_{\text{BV}} = \left(\frac{\overleftarrow{\delta}}{\delta q^a} \circ \frac{\overleftarrow{\delta}}{\delta q_a^\dagger} \right) \Big|_{(x, [\mathbf{q}], [\mathbf{q}^\dagger])}.$$

The core idea of its self-regularization roots in the congruence of attachment points for $\langle \delta \mathbf{q} |$ and $| \delta \mathbf{q}^\dagger \rangle$ in the course of evaluation of the BV-Laplacian of a functional at a section of the bundle π_{BV} (that is, for a given configuration of even physical fields ϕ^α , antighosts γ_a^\dagger etc., and for a certain choice of the antifields \mathbf{q}^\dagger ; the path integrals of $\mathcal{O}^h \cdot \exp(iS_{\text{BV}}^h/\hbar)$ in the Euler–Lagrange model at hand is in fact independent of \mathbf{q}^\dagger whenever the integrands $\mathcal{O}^h \cdot \exp(iS_{\text{BV}}^h/\hbar)$ containing the quantum BV-action S_{BV}^h satisfy the Laplace equation).

A regularization of the BV-Laplacian is often described in the literature in terms of Dirac’s δ -distributions; however, we claim that the intrinsic self-regularization scheme for Δ_{BV} is parallel

¹Indeed, the equations $\mathcal{E}_\alpha = 0$ or equations between equations $\Phi^\mu = 0$, or their higher analogs are (non)linear differential operators that map the fields ϕ^α , equations \mathcal{E}_β , etc., to sections of some vector bundles with coordinates \mathcal{E}_α , Φ^μ , etc., along the fibres. It is a Whitney sum of those bundles which we now take, obtaining the bundle π and also introducing its dual $\widehat{\pi}$ (see section 1 below).

²The traditional notation is $\mathbf{q} = \{\phi^\alpha, \gamma^{\dagger, \mu}, c_\nu^\dagger, \dots\}$ for even-parity BV-variables stemming from P_i ’s whereas $\mathbf{q}^\dagger = \{\phi_\alpha^\dagger, \gamma_\mu = (\gamma^\dagger)^\dagger_\mu, c^\nu = (c^\dagger)^{\dagger, \nu}, \dots\}$ are odd-parity BV-variables coming from the odd neighbours $\Pi \widehat{P}_i$.

to the identically-zero value of the coupling for usual covectors and vectors attached at two different points of a manifold.

This note is structured as follows. After introducing some notation and conventions, we inductively construct the space $\overline{\mathfrak{M}}^n(\pi_{\text{BV}})$ of local functionals (some of which, after a transition to the quantum setup, are the observables \mathcal{O}) by using integral functionals such as the BV-action S_{BV} as building blocks. We then recall the definition of the variational Schouten bracket \llbracket, \rrbracket as the odd Poisson bracket on the space $\overline{H}^n(\pi_{\text{BV}})$ of building blocks, then extending the antibracket to the entire linear space $\overline{\mathfrak{M}}^n(\pi_{\text{BV}})$ of local functionals by the Leibniz rule. We formulate a geometric definition of the BV-Laplacian in such a way that Δ_{BV} requires no external regularization; the operator Δ_{BV} satisfies – not just formally but in earnest – the standard product rule formula (9) that involves the variational Schouten bracket \llbracket, \rrbracket on $\overline{\mathfrak{M}}^n(\pi_{\text{BV}})$. We then pass to a more functional analytic approach, gaining a greater flexibility in dealing with variations of functionals, and we verify the main properties of the BV-Laplacian. Finally, we illustrate our reasoning in the quantum setup by a standard derivation of the quantum master equation and by a construction of the quantum BV-differential.

This note may be regarded as a continuation of the review [12]; at the same time, we further the line of reasoning from [11].

1 The space of BV-functionals

Let us first fix some notation, in most cases matching that from [12], [14] and [16] (for a more detailed exposition of these matters, see for example [14, 17, 18, 19]). Let $\pi: E \rightarrow M$ be a vector bundle of rank m over a smooth real oriented manifold of dimension n ; in this paper we assume all maps to be smooth. We let x^i be the coordinates, with indices i, j, k, \dots , along the base manifold; ϕ^α are the fibre coordinates with indices $\alpha, \beta, \gamma, \dots$.

We take the infinite jet space $\pi_\infty: J^\infty(\pi) \rightarrow M$ associated with this bundle [7, 19]; a point from the jet space is then $\theta = (x^i, \phi^\alpha, \phi_{x^i}^\alpha, \phi_{x^i x^j}^\alpha, \dots, \phi_\sigma^\alpha, \dots) \in J^\infty(\pi)$, where σ is a multi-index and we put $\phi_\emptyset^\alpha \equiv \phi^\alpha$. If $s \in \Gamma(\pi)$ is a section of π , or a *field*, we denote with $j^\infty(s)$ its infinite jet, which is a section $j^\infty(s) \in \Gamma(\pi_\infty)$. Its value at $x \in M$ is $j_x^\infty(s) = (x^i, s^\alpha(x), \frac{\partial s^\alpha}{\partial x^i}(x), \dots, \frac{\partial^{|\sigma|} s^\alpha}{\partial x^\sigma}(x), \dots) \in J^\infty(\pi)$. We denote by $\mathcal{F}(\pi)$ the ring of smooth functions on the infinite jet space; the space of top-degree horizontal forms on $J^\infty(\pi)$ is denoted by $\overline{\Lambda}^n(\pi)$. The highest horizontal cohomology, i.e., the space of equivalence classes of n -forms from $\overline{\Lambda}^n(\pi)$ modulo the image of the horizontal exterior differential \bar{d} on $J^\infty(\pi)$, is denoted by $\overline{H}^n(\pi)$; the equivalence class of $\omega \in \overline{\Lambda}^n(\pi)$ is denoted by $\int \omega \in \overline{H}^n(\pi)$. We will assume that the sections are such that integration by parts is allowed and does not result in any boundary terms (for example, the base manifold is compact, or the sections all have compact support, or decay sufficiently fast towards infinity, or are periodic). Lastly, the Euler operator is $\delta = \int d_C \cdot$, where d_C is the Cartan differential, so that the variational derivative with respect to ϕ^α is $\frac{\delta}{\delta \phi^\alpha} = \sum_{|\sigma| \geq 0} (-)^\sigma D_\sigma \frac{\partial}{\partial \phi^\alpha}$.

Let ξ be a vector bundle over $J^\infty(\pi)$, and suppose s_1 and s_2 are two sections of this bundle. They are said to be *horizontally equivalent* [10] at a point $\theta \in J^\infty(\pi)$ if $D_\sigma(s_1^\alpha) = D_\sigma(s_2^\alpha)$ at θ for all multi-indices σ and fibre-indices α . Denote the equivalence class by $[s]_\theta$. The set

$$\overline{J}_\pi^\infty(\xi) := \{[s]_\theta \mid s \in \Gamma(\xi), \theta \in J^\infty(\pi)\}$$

is called the *horizontal jet bundle* of ξ . It is clearly a bundle over $J^\infty(\pi)$, whose elements above θ are determined by all the derivatives $s_\sigma^\alpha := D_\sigma(s^\alpha)$ for all multi-indices σ and fibre-indices α .

Now suppose ζ is a bundle over M . Then the pullback bundle $\pi_\infty^*(\zeta)$ is a bundle over $J^\infty(\pi)$, so we may consider its horizontal jet bundle $\overline{J}_\pi^\infty(\pi_\infty^*(\zeta))$. (When no confusion is possible, we shall denote this by $\overline{J}_\pi^\infty(\zeta)$.) The induced bundle $\pi_\infty^*(\zeta_\infty) = \pi_\infty \times_M \zeta_\infty$ is equivalent as a bundle to $\overline{J}_\pi^\infty(\zeta) \rightarrow M$ (see e.g., [16] for a proof). This identification endows the horizontal jet space

$\overline{J_\pi^\infty}(\zeta)$ with the Cartan connection – namely the pullback connection on $\pi_\infty^*(J^\infty(\zeta))$. Therefore there exist total derivatives D_i on the horizontal jet space $\overline{J_\pi^\infty}(\zeta)$; in coordinates these are just the operators (denoting the fibre coordinate of ζ with u)³

$$D_i = \frac{\partial}{\partial x^i} + \phi_{\sigma+1_i}^\alpha \frac{\partial}{\partial \phi_\sigma^\alpha} + u_{\tau+1_i}^\beta \frac{\partial}{\partial u_\tau^\beta}, \quad 1 \leq i \leq n.$$

Thus, instead of the horizontal derivatives $D_\sigma(u^\alpha)$ of sections there are now the fibre coordinates u_σ^α , which have no derivatives along the fibre coordinates π_∞ , i.e., $\partial u_\tau^\beta / \partial \phi_\sigma^\alpha = 0$.

Let ζ be a vector bundle over M . We denote with ζ^\dagger its dual, i.e. the vector bundle whose fibre over x is the dual of the fibre of ζ over x . We also define $\widehat{\zeta} = \zeta^\dagger \otimes \Lambda^n(M)$.

If P is the space of sections of some vector bundle over $J^\infty(\pi)$ (for example, P is the space of sections that contains a certain differential equation $F \in P$, or $P = \Gamma(\pi_\infty^*(\pi)) =: \varkappa(\pi)$, to which we return on the following page), then we denote with $P^\dagger = \text{Hom}_{\mathcal{F}(\pi)}(P, \mathcal{F}(\pi))$ its dual with respect to $\mathcal{F}(\pi)$, and with $\widehat{P} = \text{Hom}_{\mathcal{F}(\pi)}(P, \overline{\Lambda}^n(\pi))$ its dual with respect to $\overline{\Lambda}^n(\pi)$. We will denote the couplings between elements of P and either P^\dagger or \widehat{P} by $\langle p', p \rangle$, which lands in either $\mathcal{F}(\pi)$ or $\overline{\Lambda}^n(\pi)$, and where p' is an element of either P^\dagger or \widehat{P} .

Returning to the vector bundle ζ over M , we see that $\pi_\infty^*(\zeta)$ is a bundle over $J^\infty(\pi)$. Set $P_\zeta = \Gamma(\pi_\infty^*(\zeta))$. Then we have that $P_\zeta^\dagger = \text{Hom}_{\mathcal{F}(\pi)}(P, \mathcal{F}(\pi)) = \Gamma(\pi_\infty^*(\zeta^\dagger))$ and $\widehat{P}_\zeta = \text{Hom}_{\mathcal{F}(\pi)}(P, \overline{\Lambda}^n(\pi)) = \Gamma(\pi_\infty^*(\widehat{\zeta}))$. Reversing the parity of the fibres [12, 15, 21] of $\widehat{\zeta}$ using the parity-reversion operator Π , we obtain the odd bundle $\Pi\widehat{\zeta}$; if the fibre coordinates of ζ are u^α , then we denote the fibre coordinates of $\Pi\widehat{\zeta}$ by u_α^\dagger (note that the coordinate index switches position). We denote by $\Pi\widehat{P}_\zeta = \Gamma(\pi_\infty^*(\Pi\widehat{\zeta}))$ the associated space of sections on $J^\infty(\pi)$.

Example 1. The space of generating sections for evolutionary derivatives is $\varkappa(\pi) = \Gamma(\pi_\infty^*(\pi)) = \Gamma(\pi) \otimes_{C^\infty(M)} \mathcal{F}(\pi)$. On the other hand, the range of the Euler operator δ is naturally isomorphic to $\widehat{\varkappa(\pi)}$ (see, e.g., [17, 18]). Therefore, since in gauge theories the relevant system of differential equations \mathcal{E} is Euler–Lagrange, i.e., $\mathcal{E} = \{\delta S = 0 \in \widehat{\varkappa(\pi)}\}$ for some $S \in \overline{\Lambda}^n(\pi)$, the space of sections P_0 that contains the differential equation $\mathcal{E} = \{\delta S = 0\}$ is $P_0 \simeq \widehat{\varkappa(\pi)}$. Applying the formalism described above to $\widehat{\varkappa(\pi)}$, we see that $\Pi\widehat{P}_0 = \Pi\varkappa(\pi) = \Gamma(\pi_\infty^*(\Pi\pi))$, and taking the horizontal jet space $\overline{J_\pi^\infty}(\widehat{\pi} \times_M \Pi\pi)$ we would obtain a jet bundle with coordinates $\phi^\alpha, \mathcal{E}_\alpha, \mathcal{E}^{\dagger, \alpha}$, and their derivatives.

Let us emphasize that the entire BV-geometry is completely determined by the initial bundle π of physical fields ϕ^α over M and by the action $S(x, [\phi])$ which yields the equations of motion and the structure of interrelations between Noether’s identities in a gauge-invariant model at hand. For consistency, we now recall the inductive construction of the BV-bundle π_{BV} over the base manifold M ; we consider the case when the total space E of the vector bundle $\pi: E \rightarrow M$ is a non-graded, purely commutative smooth real manifold (see [14, Ch. 2, 11] for more detail and also [11] where a non-graded non-commutative setup is addressed). An extension of the formalism to a \mathbb{Z}_2 - or \mathbb{Z} -graded setup is standard (specifically to the Poisson sigma-model of [6], the sign conventions are explained in [8]).

For a given bundle π and action functional $S \in \overline{H}^n(\pi)$, the m equations of motion $\mathcal{E} = \{\delta S = \mathcal{E}_\alpha \delta \phi^\alpha = 0\}$ upon ϕ^1, \dots, ϕ^m are (up to a tensor multiplication by $\overline{\Lambda}^n(\pi)$) sections of the induced bundle $\pi_\infty^*(\widehat{\pi})$ over the infinite jet bundle $J^\infty(\pi) \xrightarrow{\pi_\infty} M$; note that the dual bundle is introduced by using the $\overline{\Lambda}^n(\pi)$ -valued coupling. Indeed, each equation in the Euler–Lagrange system is a variational covector: the variations $\mathcal{E}_\alpha \delta \phi^\alpha = :d_C(S):$ are obtained after integration

³Here and in the remainder of this note we use Einstein’s summation convention: whenever an index appears twice in an expression, once as an upper index and once as a lower index, a summation over that index is implicitly present.

by parts in the equivalence class of $d_{\mathcal{L}}(S)$. Because each coefficient $\mathcal{E}_\alpha(x, [\phi])$ of $\delta\phi^\alpha$ in the left-hand side of system \mathcal{E} belongs to the horizontal module $\Gamma(\pi_\infty^*(\widehat{\pi})) = \widehat{\mathcal{X}}(\pi)$, its neighbour is the module $\Gamma(\pi_\infty^*(\Pi\pi))$ of odd-parity evolutionary vector fields. We emphasize that, as soon as the equations of motion are derived from the action, their reparametrizations $\mathcal{E}' = \mathcal{E}'(x, [\mathcal{E}])$ – yielding an equivalent but differently written system, e.g., in which two equations are swapped: $\mathcal{E}'_1 := \mathcal{E}_2 = 0$ and $\mathcal{E}'_2 := \mathcal{E}_1 = 0$ – are in principle entirely independent from a change of coordinates $\phi' = \phi'(x, [\phi])$ along the fibres of π . However, it is standard to enforce the correspondence between the fields ϕ^α and equations $\mathcal{E}_\alpha = 0$, i.e., to label the sections $\mathcal{E} \in \Gamma(\pi_\infty^*(\widehat{\pi}))$ of the bundle induced from $\widehat{\pi}$ by π_∞ by using one more time the variables ϕ^α along fibres in π instead of a rigorous use of the dual bundle $\widehat{\pi}: \widehat{E} \rightarrow M$. We thus set $\mathcal{E} \in P_0 \simeq \Gamma(\pi_\infty^*(\pi))$, which implies that $\Pi\widehat{P}_0 \simeq \Gamma(\pi_\infty^*(\Pi\widehat{\pi}))$; both isomorphisms are not canonical. The fibre coordinates in $\pi_0 := \pi$ and $\widehat{\pi}_0 = \widehat{\pi}$ in P_0 and $\Pi\widehat{P}_0$ are the physical fields ϕ^α and the antifields ϕ_α^\dagger , respectively. By this argument we construct the Whitney sum $J^\infty(\pi) \times_M J^\infty(\Pi\widehat{\pi}) \rightarrow M$ of two infinite jet bundles.⁴

Proceeding with the *linear* Noether identities $\Phi = \{\Phi^\mu(x, [\phi], [\mathcal{E}]) = 0, 1 \leq \mu \leq m_1\} \in P_1 = \Gamma(\pi_\infty \circ (\pi_\infty^*(\pi))^*(\pi_1))$, where the linearity is specific to the geometry of Euler–Lagrange equations, we recognize the auxiliary vector bundle π_1 over M such that $\Phi: \overline{J_\pi^\infty}(\pi_0) \rightarrow \Gamma(\pi_1)$ is a linear differential operator with respect to \mathcal{E}_α and may be nonlinear in ϕ^α . We conventionally denote not by Φ^μ but by $\gamma^{\dagger, \mu}$ the fibre coordinates in π_1 . Next, we construct the horizontal module $\Pi\widehat{P}_1$ of sections by employing the dual, odd-parity bundle $\widehat{\pi}_1$ with fibre coordinates γ_μ . This produces m_1 canonically conjugate ghost-antighost pairs $\gamma_\mu \leftrightarrow \gamma^{\dagger, \mu}$. As we did it at the previous step, we extend the setup further by taking the Whitney sum $J^\infty(\pi) \times_M J^\infty(\Pi\widehat{\pi}) \times_M J^\infty(\pi_1) \times_M J^\infty(\Pi\widehat{\pi}_1) \rightarrow M$ for the two generations of dual bundles.

An inductive reasoning over the finite number of higher Noether relations between the already-obtained identities gives rise to $\lambda + 1$ sets of – in total, $m + m_1 + \dots + m_\lambda$ – pairs of dual, opposite-parity bundles $\pi_i \leftrightarrow \Pi\widehat{\pi}_i$ of ranks m_i , here $0 \leq i \leq \lambda$ and $m_0 = m$. The Whitney sum

$$\pi = \bigotimes_{i=0}^{\lambda} M^{\pi_i}$$

over the base M determines the Batalin–Vilkovisky (BV) superbundle

$$\pi_{\text{BV}} = \pi_\infty^*(\Pi\widehat{\pi}_\infty): \overline{J_\pi^\infty}(\Pi\widehat{\pi}) = J^\infty(\pi) \times_M J^\infty(\Pi\widehat{\pi}) \longrightarrow M.$$

We denote by

$$\mathbf{q} = (\phi, \gamma^\dagger, c^\dagger, \dots)$$

the entire set of even-parity BV-coordinates and by

$$\mathbf{q}^\dagger = (\phi^\dagger, \gamma, c, \dots)$$

their odd-parity duals; the convention $(\gamma^\dagger)^\dagger = \gamma$ resolves a minor confusion in the usually accepted notation.

Remark 1. When speaking of parities $(-)^{(\cdot)}$, we refer to the *ghost numbers* $\text{gh}(\cdot)$ and the respective parity reversion operator Π (for instance, $\text{gh}(\phi^\alpha) = 0$, $\text{gh}(\phi_\alpha^\dagger) = 1$, etc.); the entire collection of auxiliary bundles – from π_0 for P_0 or π_1 for P_1 to π_λ for P_λ – is even with respect

⁴A rigorous approach to the correspondence between fields and Euler–Lagrange equations, which we tracked in Example 1 above, would give us the horizontal jet bundles with fibre coordinates $\mathcal{E}_{\alpha, \sigma}$ and $\mathcal{E}_\tau^{\dagger, \beta}$ over $J^\infty(\pi)$, here $|\sigma|, |\tau| \geq 0$.

to the ghost parity. The ghost parities of objects that belong to P_i and $\Pi\widehat{P}_i$ at each i are always opposite: the former are ghost-parity even and the latter are odd. Should one consider an independently graded setup of fields ϕ^α as fibres of a superbundle $\pi = (\pi^0|\pi^1)$ or a setup with \mathbb{Z} -graded differential forms taken as fields (see [6]), and should one then introduce the action functional $S \in \overline{H}^n(\pi)$, a count of extra signs in the equations of motion, Noether identities etc., would be tedious but straightforward (c.f. Example 3 on page 12).

Note further that the choice of canonically conjugate variables of opposite ghost parities is rigid in the sense that it is completely determined by the action functional S : one can not take ϕ_α^\dagger 's as new, odd-parity ‘fields’ and let the old unknowns ϕ^α be the conjugate ‘antifields’ (although $\Pi \circ \Pi = \text{id}$ and $\widehat{\widehat{P}}_i = P_i$ in finite dimension) because the trivial equations of motion $\delta S / \delta \phi^\dagger \equiv 0$ do not constrain the physical fields.

Remark 2. A division of the BV-coordinates $\mathbf{q} \leftrightarrow \mathbf{q}^\dagger$ by using the *vector*⁵ bundles $\pi_0, \dots, \pi_\lambda$ leads to the understanding of their variations $\langle \delta \mathbf{q} |$ and $|\delta \mathbf{q}^\dagger \rangle$ as covectors and odd-parity vectors, respectively (see [1, §37]). At the same time, the following heuristic argument motivates a choice –in what follows, irrelevant– of an upper or lower location of the index that enumerates fibre components in the bundle π_i or $\Pi\widehat{\pi}_i$ at fixed i . Namely, we remember that ϕ^α is a fibre coordinate in the initial vector bundle π , whence the odd-parity antifields ϕ_β^\dagger along the fibre of $\Pi\widehat{\pi}$ can be viewed as covectors. Therefore, the variation $\langle \delta \phi^\alpha |$ is indeed a covector and $|\delta \phi_\beta^\dagger \rangle$ automatically becomes a vector. Next, let us recall that the Euler–Lagrange equations of motion $\mathcal{E} = \{\delta S = \mathcal{E}_\alpha \delta \phi^\alpha = 0\} \in \widehat{\mathcal{X}}(\pi)$ are variational covectors by construction. The first generation of linear Noether’s relations $\Phi = \{\Phi^\mu(x, [\phi^\alpha], [\mathcal{E}_\beta]) = 0\} \in P_1$ between the equations of motion $\mathcal{E} \in P_0$ is determined by linear differential operators in total derivatives acting on the components of the variational covector $\mathcal{E} = \delta S$. Consequently, the antighosts $\gamma^{\dagger, \alpha}$ are even vectors and their parity-reversed duals γ_a are odd covectors, so that $\langle \delta \gamma^{\dagger, \alpha}, \delta \gamma_\beta \rangle = +1 \cdot \delta_\beta^\alpha$. It is similar for the second generation of ghost-antighost pairs $c^\alpha \leftrightarrow c_\beta^\dagger$, etc. (see [14, Ch. 2 and 11]).

We finally notice that the fact that an object may be a variational (co)vector relative to the initial vector bundle π of physical fields is not necessarily correlated with the ghost parity.

We now describe the space of BV-functionals. First, we denote by $\overline{H}^n(\pi_{\text{BV}})$ the space of equivalence classes (modulo the image of horizontal differential \overline{d}) of top-degree horizontal forms on $\overline{J}_\pi^\infty(\Pi\widehat{\pi})$; in other words, elements $\mathcal{H} \in \overline{H}^n(\pi_{\text{BV}})$ are integral functionals possibly depending on the entire set of BV-variables \mathbf{q} and \mathbf{q}^\dagger and their derivatives $\mathbf{q}_\sigma, \mathbf{q}_\tau^\dagger$ of arbitrarily high but finite orders (by convention, $\mathbf{q}_\emptyset \equiv \mathbf{q}, \mathbf{q}_\emptyset^\dagger \equiv \mathbf{q}^\dagger$ and $0 \leq |\sigma|, |\tau| < \infty$). The full BV-action $S_{\text{BV}} = S + \dots$ for a given gauge-invariant model is a standard example of such integral functional.

Every equivalence class $\mathcal{H} = \int h(x, [\mathbf{q}], [\mathbf{q}^\dagger]) d^n x \in \overline{H}^n(\pi_{\text{BV}})$ of the highest horizontal cohomology group for the vector bundle π_{BV} determines a map $\mathcal{H}: \Gamma(\pi_{\text{BV}}) \rightarrow \mathbb{k}$ (which we let be either \mathbb{R} or, possibly, \mathbb{C}). Namely, for any $s \in \Gamma(\pi_{\text{BV}})$ we set

$$\mathcal{H}: s \mapsto \mathcal{H}(s) = \int_M j^\infty(s)^* h(x, [\mathbf{q}], [\mathbf{q}^\dagger]) d^n x \in \mathbb{k}. \quad (1)$$

The BV-action S_{BV} is a typical mapping that takes sections to numbers. We employ such integral functions from $\overline{H}^n(\pi_{\text{BV}})$ as building blocks in the construction of a larger set of ‘admissible’ functionals which also map $\Gamma(\pi_{\text{BV}})$ to the field \mathbb{k} . In agreement with the ideology of [17, 18], let us view sections $s \in \Gamma(\pi_{\text{BV}})$ as “points” and cohomology classes from $\overline{H}^n(\pi_{\text{BV}})$ as particular examples of \mathbb{k} -valued “functions” defined at every “point” of the space of sections.⁶ Next, we

⁵Note that the *vector* bundle π is used to introduce the physical fields ϕ^1, \dots, ϕ^m yet these objects could be, for example, components of a *covector* $A = \sum_{\alpha=1}^n \phi^\alpha dx^\alpha$ of a gauge connection’s one-form in a principal fibre bundle over M^n with a given structure Lie group (see Example 2 on page 11).

⁶Note that all highest horizontal cohomology group elements are such “functions”, but not all “functions” are integral functionals.

introduce the multiplicative structure $F \otimes_{\mathbb{k}} G \mapsto F \cdot G$ pointwise by the rule

$$(F \cdot G)(s) \stackrel{\text{def}}{=} F(s) \cdot_{\mathbb{k}} G(s), \quad s \in \Gamma(\pi_{\text{BV}}),$$

for two integral functionals $F, G \in \overline{H}^n(\pi_{\text{BV}})$.

By taking (formal) sums of products of arbitrary finite numbers of integral functionals from $\overline{H}^n(\pi_{\text{BV}})$, i.e., by viewing such functionals as building blocks, we generate the linear subspace,

$$\overline{\mathfrak{M}}^n(\pi_{\text{BV}}) = \bigoplus_{i=1}^{+\infty} \bigotimes_{\mathbb{k}}^i \overline{H}^n(\pi_{\text{BV}}) \subseteq \mathfrak{M}^n(\pi_{\text{BV}}) = \text{Map}_{\mathbb{k}}(\Gamma(\pi_{\text{BV}}) \rightarrow \mathbb{k}), \quad (2)$$

in the space $\mathfrak{M}^n(\pi_{\text{BV}})$ of *all* maps which assign a number to every section. We do not claim that the subspace $\overline{\mathfrak{M}}^n(\pi_{\text{BV}})$ of local functionals exhausts the entire set of existing mappings; we emphasize that in what follows we describe the construction of the Schouten bracket and BV-Laplacian on that linear subspace $\overline{\mathfrak{M}}^n(\pi_{\text{BV}}) \subseteq \mathfrak{M}^n(\pi_{\text{BV}})$ (that is, not just on the set of equivalence classes $\overline{H}^n(\pi_{\text{BV}})$, still not for arbitrary maps in $\mathfrak{M}^n(\pi_{\text{BV}})$). The reason why we do so is that integration by parts makes sense for each constituent term from $\overline{H}^n(\pi_{\text{BV}})$ of a composite functional from $\overline{\mathfrak{M}}^n(\pi_{\text{BV}})$.

Remark 3. By constructing the space of maps $\overline{\mathfrak{M}}^n(\pi_{\text{BV}})$ in this fashion, we have not completely exited the realm of integral functionals on jet spaces. To see this, take two integral functionals $F = \int f(x, [\mathbf{q}], [\mathbf{q}^\dagger]) d^n x$ and $G = \int g(x', [\mathbf{q}'], [\mathbf{q}'^\dagger]) d^n x'$ on π_{BV} . Then we may write $F \cdot G$ as $F \cdot G = \int f(x, [\mathbf{q}], [\mathbf{q}^\dagger]) g(x', [\mathbf{q}'], [\mathbf{q}'^\dagger]) d^n x \wedge d^n x'$, which shows that $F \cdot G \in \overline{H}^{2n}(\pi_{\text{BV}} \times \pi_{\text{BV}})$, which is the top level cohomology of the jet space of the product bundle $\pi_{\text{BV}} \times \pi_{\text{BV}}$. Thus, $F \cdot G$ is an integral functional with respect to the bundle $\pi_{\text{BV}} \times \pi_{\text{BV}}$, and we see that the second term $\overline{H}^n(\pi_{\text{BV}}) \otimes_{\mathbb{k}} \overline{H}^n(\pi_{\text{BV}})$ in the direct sum in (2) is a subspace of $\overline{H}^{2n}(\pi_{\text{BV}} \times \pi_{\text{BV}})$. Continuing this line of reasoning to products of any number of factors we find

$$\overline{\mathfrak{M}}^n(\pi_{\text{BV}}) \subseteq \bigoplus_{i=1}^{+\infty} \overline{H}^{i \cdot n}(\underbrace{\pi_{\text{BV}} \times \cdots \times \pi_{\text{BV}}}_{i \text{ copies}}).$$

(Note that this is a strict inclusion; for example, $\overline{H}^{2n}(\pi_{\text{BV}} \times \pi_{\text{BV}})$ contains functionals such as $\int q_{xx'} d^n x \wedge d^n x'$ while $\overline{H}^n(\pi_{\text{BV}}) \otimes_{\mathbb{k}} \overline{H}^n(\pi_{\text{BV}})$ does not.) Thus, although composite but homogeneous functionals from $\overline{\mathfrak{M}}^n(\pi_{\text{BV}})$ (i.e., functionals of the form $F = F_1 \cdots F_i$ for some i and $F_k \in \overline{H}^n(\pi)$ for $1 \leq k \leq i$) are not integral with respect to the bundle π_{BV} , they *are* integral with respect to a bundle, namely $\underbrace{\pi_{\text{BV}} \times \cdots \times \pi_{\text{BV}}}_{i \text{ copies}}$.

2 The Schouten bracket

Consider first the space $\overline{H}^n(\pi_{\text{BV}})$ of integral functionals, that is, equivalence classes of highest-degree horizontal forms modulo exact forms in the image of $\bar{d} = \pi_{\text{BV}}^*(d_{\text{dR}}(M))$. We are dealing with a geometry in which densities of integral functionals can depend on the (anti)fields and all the available generations of (anti)ghosts, and their derivatives. Let us remember that the “anti-objects” q_α^\dagger are introduced as the duals of the BV-vector coordinates q^β so that $\langle \delta q^\alpha, \delta q_\beta^\dagger \rangle = +1 \cdot \delta_\beta^\alpha$ and $\langle \delta q_\beta^\dagger, \delta q^\alpha \rangle = -1 \cdot \delta_\beta^\alpha$ are the \mathbb{k} -valued couplings of the dual bases⁷ (obviously, δ_β^α is the Kronecker symbol, not a variation).

⁷At this point, the appearance of the minus sign – whenever the variations are swapped – is logical because the Cartan differentials must anticommute under the wedge product. On the other hand, this sign change will re-appear in sections 4–5, taking the shape of a choice of the dual bases in fibres of the bundle $\pi \times_M \Pi\widehat{\pi}$: in $\Pi(\Pi\widehat{\pi})$, the basis of dual vectors to a frame in $\Pi\widehat{\pi}$ is *minus* the initially taken basis in a fibre of π .

To explain the minus sign in the second coupling, let us recall that the variations $\delta(\cdot)$ of jet variables are shorthand notation for a projection of the vertical Cartan differential, which acts –along the fibres of π_∞ – onto the horizontal cohomology with respect to the horizontal differential \bar{d} . Consequently, the geometry is \mathbb{Z}_2 -graded by the parity-reversion operator Π , which is introduced by hands to produce Manin’s odd neighbours (c.f. [21] and [12, 15]), and it is also \mathbb{Z} -graded by the degree of Cartan forms (this degree equals zero for the action $S \in \overline{H}^n(\pi)$, BV-action $S_{\text{BV}} \in \overline{H}^n(\pi_{\text{BV}})$, and generally, for all elements of $\overline{H}^n(\pi_{\text{BV}})$ – but equals one for the variation δS_{BV} , etc.). On top of that, the initial bundle π itself can be graded by a suitable ring (usually, it is \mathbb{Z}_2 again so that $\pi = (\pi^{\bar{0}}|\pi^{\bar{1}})$ and these parities of the fibre variables in π are $\varepsilon_\alpha \in \mathbb{Z}_2$). For the sake of clarity we assume onwards that the bundle π of physical fields is purely even; nevertheless, the variation symbols δ anticommute between themselves, independently from the ghost-parity Π that distinguishes between q^α and q^\dagger_β . We let $\delta q^\alpha \wedge \delta q^\dagger_\beta$ be the odd-parity symplectic form on $J^\infty(\pi) \times_M J^\infty(\Pi\hat{\pi})$.

Definition 1. Let $F, G \in \overline{H}^n(\pi_{\text{BV}})$ be two integral functionals depending, generally speaking, on the entire collection of BV-variables and their derivatives along the base M of arbitrary high but finite order. The *variational Schouten bracket* of the building blocks F and G is

$$\llbracket F, G \rrbracket = \langle \overrightarrow{\delta F} \wedge \overleftarrow{\delta G} \rangle, \quad (3)$$

where the arrows indicate the direction in which, with due attention to the \mathbb{Z}_2 -parity brought in by Π , the variations of BV-variables \mathbf{q} and \mathbf{q}^\dagger are transported. After this transportation they are coupled, as indicated by the angular brackets $\langle \cdot \rangle$.

Remark 4. This construction of the odd Poisson bracket $\llbracket \cdot, \cdot \rrbracket : \overline{H}^n(\pi_{\text{BV}}) \times \overline{H}^n(\pi_{\text{BV}}) \rightarrow \overline{H}^n(\pi_{\text{BV}})$ is standard (see [14, Ch. 8] for its extension to a noncommutative setup of cyclic-invariant words); let us now focus on the mechanism of the ordered coupling between the initially separate geometries for F and G . Namely, as was explained in Remark 3, each of these two factors is defined over its own copy of the bundle $\overline{J}_\pi^\infty(\Pi\hat{\pi}) \rightarrow J^\infty(\pi) \rightarrow M$ with all canonically conjugate pairs of BV-variables \mathbf{q} and \mathbf{q}^\dagger and their derivatives \mathbf{q}_σ and \mathbf{q}^\dagger_τ for coordinates along the fibres over points of the base manifold M . In each of the two copies we vary, that is, we take the vertical differential d_C and integrate by parts.⁸ Since F and G are defined over separate copies of π_{BV} , the integration by parts on the one does not feel the other:

$$\begin{aligned} \llbracket F, G \rrbracket &= \langle \overrightarrow{\delta F} \wedge \overleftarrow{\delta G} \rangle = \left\langle \int \overrightarrow{d_C}(f(x, [\mathbf{q}], [\mathbf{q}^\dagger])) d^n x \wedge \int \overleftarrow{d_C}(g(y, [\mathbf{q}'], [\mathbf{q}'^\dagger])) d^n y \right\rangle \\ &= \left\langle \iint \left(\frac{\overrightarrow{\delta f}}{\delta q^\alpha} \delta q^\alpha + \frac{\overrightarrow{\delta f}}{\delta q^\dagger_\alpha} \delta q^\dagger_\alpha \right) \Big|_{(x, [\mathbf{q}], [\mathbf{q}^\dagger])} \wedge \left(\delta q^\alpha \frac{\overleftarrow{\delta g}}{\delta q^\alpha} + \delta q^\dagger_\alpha \frac{\overleftarrow{\delta g}}{\delta q^\dagger_\alpha} \right) \Big|_{(y, [\mathbf{q}'], [\mathbf{q}'^\dagger])} d^n x d^n y \right\rangle. \end{aligned}$$

Now begins the merging of the two bundles: we first identify their bases M , then the fibres of π , and finally we identify both copies of the BV-bundle π_{BV} . It is the coupling $\langle \cdot, \cdot \rangle$ of covectors and vectors which provides the identification of points in the mechanism of merging. Indeed, the pairs of coupling-dual Cartan differentials from $\overrightarrow{\delta F}$ and $\overleftarrow{\delta G}$, whenever attached to *different* points of the space $\overline{J}_\pi^\infty(\Pi\hat{\pi})$, couple to zero, whereas the same coupling yields $\pm 1 \in \mathbb{k}$ if these points coincide (this is identical to the coupling $T_y^*N \times T_x N \rightarrow \mathbb{R}$ for any smooth manifold N).

This scenario can be recognized also as Dirac’s δ -distribution $\delta(x-y)$, in which y runs over the base M and which integrates away⁹ the n differentials $d^n y$ in one of the two integral functionals,

⁸The value of the Schouten bracket $\llbracket F, G \rrbracket$ does not depend on the choice of densities in both factors because the variations of exact forms are equal to zero by our earlier convention of the absence of boundary terms.

⁹The convention that the Schouten bracket of two integral functionals remains an integral functional determines a natural choice of the \mathbb{k} -valued coupling $\langle \cdot, \cdot \rangle$ by using the variables \mathbf{q}^\dagger rather than a choice of a $\overline{H}^n(\pi_{\text{BV}})$ -valued coupling by using the metric-dependent Hodge structure $*$ on M and the conjugate variables \mathbf{q}^* , see, e.g., [6].

$F = \int f \, d^n x$ and $G = \int g \, d^n y$. Hence we conclude that the contributions from F and G to their bracket $\llbracket F, G \rrbracket$ are evaluated at the (infinite jet $j^\infty(s)$ of the) same section $s \in \Gamma(\pi_{\text{BV}})$.

Corollary 1. *In coordinates, the Schouten bracket $\llbracket F, G \rrbracket$ of two functionals $F = \int f \, d^n x$, $G = \int g \, d^n y \in \overline{H}^n(\pi_{\text{BV}})$ is the integral functional given by the formula*

$$\llbracket F, G \rrbracket = \int \left(\frac{\overrightarrow{\delta f}}{\delta q^\alpha} \cdot \frac{\overleftarrow{\delta g}}{\delta q_\alpha^\dagger} - \frac{\overrightarrow{\delta f}}{\delta q_\alpha^\dagger} \cdot \frac{\overleftarrow{\delta g}}{\delta q^\alpha} \right) d^n x,$$

where the arguments of all the differential functions coincide and are equal to $\theta^\infty = (x, [\mathbf{q}], [\mathbf{q}^\dagger]) \in \overline{J_\pi^\infty}(\Pi\hat{\pi})$.

Finally, we extend the Schouten bracket from the space $\overline{H}^n(\pi_{\text{BV}})$ of building blocks (1) to the linear subspace $\overline{\mathfrak{M}}^n(\pi_{\text{BV}})$ generated in $\mathfrak{M}^n(\pi_{\text{BV}})$ by using elements from $\overline{H}^n(\pi_{\text{BV}})$ under the multiplication in \mathbb{k} for their values at sections $s \in \Gamma(\pi_{\text{BV}})$. We take the following theorem as the recursive definition.

Theorem 1. *The geometric construction of $\llbracket \cdot, \cdot \rrbracket$ in Definition 1 extends to the antibracket $\llbracket \cdot, \cdot \rrbracket$ on $\overline{\mathfrak{M}}^n(\pi_{\text{BV}})$ such that the Leibniz rule*

$$\begin{aligned} \llbracket F, G \cdot H \rrbracket(s) &= (\llbracket F, G \rrbracket \cdot H)(s) + (-)^{(\text{gh}(F)-1) \cdot \text{gh}(G)} (G \cdot \llbracket F, H \rrbracket)(s) \\ &= \llbracket F, G \rrbracket(s) \cdot H(s) + (-)^{(\text{gh}(F)-1) \cdot \text{gh}(G)} G(s) \cdot \llbracket F, H \rrbracket(s) \end{aligned}$$

holds for any $F, G, H \in \overline{\mathfrak{M}}^n(\pi_{\text{BV}})$ and all $s \in \Gamma(\pi_{\text{BV}})$. The antibracket $\llbracket \cdot, \cdot \rrbracket$ on $\overline{\mathfrak{M}}^n(\pi_{\text{BV}})$ is bi-linear over \mathbb{k} , shifted-graded skew-symmetric,

$$\llbracket F, G \rrbracket = -(-)^{(\text{gh}(F)-1)(\text{gh}(G)-1)} \llbracket G, F \rrbracket,$$

and satisfies the shifted-graded Jacobi identity:¹⁰

$$\begin{aligned} 0 &= (-)^{(\text{gh}(F)-1)(\text{gh}(H)-1)} \llbracket F, \llbracket G, H \rrbracket \rrbracket + (-)^{(\text{gh}(F)-1)(\text{gh}(G)-1)} \llbracket G, \llbracket H, F \rrbracket \rrbracket \\ &\quad + (-)^{(\text{gh}(G)-1)(\text{gh}(H)-1)} \llbracket H, \llbracket F, G \rrbracket \rrbracket. \end{aligned} \quad (4)$$

Proof. For simplicity, we assume that $F \in \overline{H}^n(\pi_{\text{BV}})$ is an integral functional. Since the bracket is shifted-graded skew-symmetric, any reasoning that applies to the right slot of the bracket will also hold for the left slot, so no generality is lost. We also assume, again without loss of generality, that G is also an integral functional, and that H is composite but homogeneous (i.e. $H = H_1 \cdots H_i$, where the factors are elements of $\overline{H}^n(\pi_{\text{BV}})$). By realizing that $H = H_1 \cdot H_2 \cdots H_i$ is also a product of one integral functional with several others, one may continue the following process inductively to reduce the bracket of F and $G \cdot H$ to an expression which contains only brackets with integral functionals on π_{BV} as arguments.

Following Remark 3, we consider $G \cdot H$ to be an integral functional with respect to the product bundle

$$\pi_{\text{BV}} \times \underbrace{\pi_{\text{BV}} \times \cdots \times \pi_{\text{BV}}}_{i \text{ copies}} = \pi_{\text{BV}}^{(0)} \times \pi_{\text{BV}}^{(1)} \times \cdots \times \pi_{\text{BV}}^{(i)},$$

of $1 + i$ copies of the BV-bundle π_{BV} . The total space of this product of bundles is endowed with the Cartan differential

$$d_C = d_C^{(0)} + d_C^{(1)} + \cdots + d_C^{(i)},$$

¹⁰For a proof of these last three standard properties (namely, bi-linearity, etc.) of the variational Schouten bracket in the (non)commutative setup see, e.g., [14, Ch. 8], [11] and [16].

where the superscripts indicate from which copy of π_{BV} the operator comes. Note that the integration by parts, whenever one transforms $d_C^{(a)} \mapsto \delta'$ or $d_C^{(a)} \mapsto \delta''$ inside the a th building block $\overline{H}^n(\pi)$ within G or H , does not produce any effect on the other i factors because their geometries are entirely independent from the geometry of π_{BV} . Consequently, all total derivatives from $\pi_{\text{BV}}^{(a)}$ falling outside $\overline{H}^n(\pi_{\text{BV}}^{(a)})$ evaluate to zero, whence

$$\overleftarrow{\delta}(H) = \sum_{a=1}^{i'} H_1 \cdots \overleftarrow{\delta}(H_a) \cdots H_{i'},$$

where the resulting variations δq^α and δq_α^\dagger are to be transported all the way to the left through $H_1 \cdots H_{a-1}$, as indicated by the arrow on top of the Euler operator δ . Summarizing, we have

$$\overleftarrow{\delta}(G \cdot H) = \overleftarrow{\delta'} G \cdot H + G \cdot \overleftarrow{\delta''} H.$$

Then the Schouten bracket is given by equation (3) (suppressing the suffix (s) which indicates that everything is evaluated on a section s):

$$\llbracket F, G \cdot H \rrbracket = \langle \overrightarrow{\delta F} \wedge \overleftarrow{\delta}(G \cdot H) \rangle = \langle \overrightarrow{\delta F} \wedge \overleftarrow{\delta'} G \cdot H \rangle + \langle \overrightarrow{\delta F} \wedge G \cdot \overleftarrow{\delta''} H \rangle$$

In the right term, the differentials have to be transported to the left before the coupling can happen. In order to achieve this we swap G with $\overleftarrow{\delta''} H$, obtaining a minus sign $(-)^{\text{gh}(G)\text{gh}(H)}$:

$$= \langle \overrightarrow{\delta F} \wedge \overleftarrow{\delta'} G \cdot H \rangle + (-)^{\text{gh}(G)\text{gh}(H)} \langle \overrightarrow{\delta F} \wedge \overleftarrow{\delta''} H \rangle \cdot G$$

Swapping the factors in the right term, we obtain:

$$\begin{aligned} &= \langle \overrightarrow{\delta F} \wedge \overleftarrow{\delta'} G \rangle \cdot H + (-)^{\text{gh}(G)\text{gh}(H)} (-)^{(\text{gh}(F)+\text{gh}(H)-1)\text{gh}(G)} G \cdot \langle \overrightarrow{\delta F} \wedge \overleftarrow{\delta''} H \rangle \\ &= \langle \overrightarrow{\delta F} \wedge \overleftarrow{\delta'} G \rangle \cdot H + (-)^{(\text{gh}(F)-1)\text{gh}(G)} G \cdot \langle \overrightarrow{\delta F} \wedge \overleftarrow{\delta''} H \rangle \\ &= \llbracket F, G \rrbracket \cdot H + (-)^{(\text{gh}(F)-1)\text{gh}(G)} G \cdot \llbracket F, H \rrbracket. \end{aligned}$$

This proves the claim. ■

3 The BV-Laplacian

The BV-Laplacian Δ_{BV} is a specific linear operator on the linear subspace $\overline{\mathfrak{M}}^n(\pi_{\text{BV}}) \subseteq \mathfrak{M}^n(\pi_{\text{BV}})$ of those functionals $\mathcal{H}: \Gamma(\pi_{\text{BV}}) \rightarrow \mathbb{k}$ which are assembled by using (sums of products of) the integral functionals from $\overline{H}^n(\pi_{\text{BV}})$. As in the previous section, we start with the definition of Δ_{BV} on the space $\overline{H}^n(\pi_{\text{BV}})$ of elementary building blocks.

Definition 2 (provisional, c.f. §4). The *BV-Laplacian* $\Delta_{\text{BV}}: \overline{H}^n(\pi_{\text{BV}}) \rightarrow \overline{H}^n(\pi_{\text{BV}})$ on the space of integral functionals – possibly, depending on the entire collection of BV-variables and on their derivatives up to arbitrarily high order – is the linear mapping

$$\Delta_{\text{BV}}: \mathcal{H} \mapsto \frac{1}{2} \langle \overleftarrow{\delta} * \overleftarrow{\delta}(\mathcal{H}) \rangle, \quad \mathcal{H} \in \overline{H}^n(\pi_{\text{BV}}), \quad (5)$$

where $\langle \rangle$ denotes the ordered \mathbb{k} -valued coupling $\langle \delta q^\alpha \wedge \delta q_\beta^\dagger \rangle = \delta_\beta^\alpha = -\langle \delta q^\alpha \wedge \delta q_\beta^\dagger \rangle$ of Cartan differentials for the pairs $q^\alpha \leftrightarrow q_\alpha^\dagger$ of canonically conjugate variables provided by π_{BV} , and where the operator $*$ is $(-)^{\text{gh}(\cdot)-1}$ is such that $*(\delta q^\alpha) = -\delta q^\alpha$ and $*(\delta q_\beta^\dagger) = \delta q_\beta^\dagger$.

Corollary 2. *In coordinates, the BV-Laplacian Δ_{BV} acts on an integral functional $\mathcal{H} = \int h(x, [q], [q^\dagger]) d^n x \in \overline{H}^n(\pi_{\text{BV}})$ by the formula*

$$\Delta_{\text{BV}}(\mathcal{H}) = \int \left(\frac{\overleftarrow{\delta}}{\delta q^\alpha} \left(\frac{\overleftarrow{\delta} h}{\delta q_\alpha^\dagger} \right) \right) \Big|_{(x, [q], [q^\dagger])} d^n x, \quad (6)$$

which thus yields an integral functional again: $\Delta_{\text{BV}}(\mathcal{H}) \in \overline{H}^n(\pi_{\text{BV}})$.

Proof. By the construction of BV-variables q and q^\dagger we have that $\overleftarrow{\delta} = \overleftarrow{\delta}_q + \overleftarrow{\delta}_{q^\dagger}$. Note that the effect of the operator $*$ is to turn δq into $-\delta q$, while it maps δq^\dagger to itself. Then

$$\frac{1}{2} \langle \overleftarrow{\delta} * \overleftarrow{\delta}(\mathcal{H}) \rangle = \frac{1}{2} \langle (\overleftarrow{\delta}_q + \overleftarrow{\delta}_{q^\dagger})(-\overleftarrow{\delta}_q + \overleftarrow{\delta}_{q^\dagger})(\mathcal{H}) \rangle = \frac{1}{2} \langle (-\overleftarrow{\delta}_q^2 + \overleftarrow{\delta}_q \overleftarrow{\delta}_{q^\dagger} - \overleftarrow{\delta}_{q^\dagger} \overleftarrow{\delta}_q - \overleftarrow{\delta}_{q^\dagger}^2)(\mathcal{H}) \rangle$$

Since δq and δq^\dagger are differentials, the leftmost and rightmost terms vanish; moreover, they anticommute, so the second and third term collapse onto each other:¹¹

$$= \langle \overleftarrow{\delta}_q (\overleftarrow{\delta}_{q^\dagger}(\mathcal{H})) \rangle = \left\langle \overleftarrow{\delta}_q \left(\int \delta q_\alpha^\dagger \frac{\overleftarrow{\delta} h}{\delta q_\alpha^\dagger} d^n x \right) \right\rangle = \left\langle \int \delta q^\beta (-)^{|\tau|} D_\tau \left(\delta q_\alpha^\dagger \frac{\overrightarrow{\partial}}{\partial q_\tau^\beta} \frac{\overleftarrow{\delta} h}{\delta q_\alpha^\dagger} \right) d^n x \right\rangle. \quad (7)$$

Some of the derivatives will fall on δq_α^\dagger , while the others fall on what stands to the right of δq_α^\dagger . Let us recall, however, that the \mathbb{k} -valued coupling from $\pi_\infty^*(\Pi \widehat{\pi}_\infty)$ for the variations of canonically conjugate variables is such that

$$\begin{aligned} \langle d_C q_\tau^\beta, d_C q_{\alpha, \sigma}^\dagger \rangle &= (-)^{\text{gh}(q^\alpha)} \delta_\beta^\alpha \cdot \delta_{\sigma \cup \tau}^\varnothing, \\ \langle d_C q_{\alpha, \sigma}^\dagger, d_C q_\tau^\beta \rangle &= -(-)^{\text{gh}(q^\alpha)} \delta_\beta^\alpha \cdot \delta_{\sigma \cup \tau}^\varnothing, \end{aligned}$$

where the second Kronecker delta-symbol is nonzero only if both multi-indexes σ and τ are empty. This definition selects only those terms in which *all* the derivatives fall on the coefficients of the vertical two-form. Thus, we obtain

$$\frac{1}{2} \langle \overleftarrow{\delta} * \overleftarrow{\delta}(\mathcal{H}) \rangle = \int \langle \delta q^\beta \wedge \delta q_\alpha^\dagger \rangle \left(\frac{\overleftarrow{\delta}}{\delta q^\beta} \left(\frac{\overleftarrow{\delta} h}{\delta q_\alpha^\dagger} \right) \right) \Big|_{(x, [q], [q^\dagger])} d^n x,$$

which is the well-known expression (6) from the literature. ■

Example 2. Take a compact, semisimple Lie group G with Lie algebra \mathfrak{g} and consider the corresponding Yang-Mills theory. Write A_i^a for the (coordinate expression of) the gauge potential A – a lower index i because A is a one-form on the base manifold (i.e., a covector), and an upper index a because A is a vector in the Lie algebra \mathfrak{g} of the Lie group G . Defining the field strength \mathcal{F} by $\mathcal{F}_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + f_{bc}^a A_i^b A_j^c$ where f_{bc}^a are the structure constants of \mathfrak{g} , the Yang-Mills action is

$$S_0 = \frac{1}{4} \int \mathcal{F}_{ij}^a \mathcal{F}^{a, ij} d^n x,$$

and the full BV-action S_{BV} is

$$S_{\text{YM}} = S_0 + \int A_a^{i\dagger} (D_i \gamma^a + f_{bc}^a A_i^b \gamma^c) d^4 x - \frac{1}{2} \int f_{ab}^c \gamma^a \gamma^b \gamma_c^\dagger d^4 x.$$

¹¹In our conventions, the left arrow on $\overleftarrow{\delta}$ means that the variations δq_α^\dagger are to be pushed to the left. As a consequence of this, the operator $\overleftarrow{\delta} / \delta q_\alpha^\dagger$ acts from the left; in the literature this is usually denoted with an arrow pointing towards the *right*. Therefore the partial derivatives $\overrightarrow{\partial} / \partial q_{\alpha, \sigma}^\dagger$ that occur in $\overleftarrow{\delta} / \delta q_\alpha^\dagger$ have reversed arrows.

Let us calculate the BV-Laplacian of this functional. As a consequence of equation (6), the only terms which survive in $\Delta_{\text{BV}}(S_{\text{YM}})$ are those which contain both A and A^\dagger , or both γ and γ^\dagger . Therefore,

$$\begin{aligned}\Delta_{\text{BV}}(S_{\text{YM}}) &= \int \left(\frac{\overleftarrow{\delta}}{\delta A_j^d} \frac{\overleftarrow{\delta}}{\delta A_d^{j\dagger}} (f_{bc}^a A_a^{i\dagger} A_i^b \gamma^c) - \frac{1}{2} \frac{\overleftarrow{\delta}}{\delta \gamma_d^\dagger} \frac{\overleftarrow{\delta}}{\delta \gamma^d} (f_{ab}^c \gamma^a \gamma^b \gamma_c^\dagger) \right) d^4x \\ &= \int \left(\frac{\overleftarrow{\delta}}{\delta A_j^d} (f_{bc}^d A_j^b \gamma^c) - \frac{1}{2} \frac{\overleftarrow{\delta}}{\delta \gamma_d^\dagger} (f_{db}^c \gamma^b \gamma_c^\dagger - f_{ad}^c \gamma^a \gamma_c^\dagger) \right) d^4x \\ &= \int \left(f_{dc}^d \gamma^c - \frac{1}{2} (f_{db}^d \gamma^b + f_{ad}^d \gamma^a) \right) d^4x = 0.\end{aligned}$$

Since the BV-action S_{YM} is by construction such that $\llbracket S_{\text{YM}}, S_{\text{YM}} \rrbracket$ is zero, it follows that S_{YM} satisfies the quantum master equation (23) tautologically – both sides are, by independent calculations, equal to zero.

Example 3. Consider the nonlinear Poisson sigma model introduced in [6]. Since its fields are not all purely even, we would have to generalize all of our reasoning so far to a \mathbb{Z}_2 -graded setup – which is, as noted before, tedious but straightforward. A calculation of $\Delta_{\text{BV}}(S_{\text{CF}})$ of the BV-action S_{CF} of this model would, up to minor differences in conventions and notations, proceed just as it does in the paper itself, in section 3.2 – except that no infinite constants or delta functions appear.

Proposition 1. *The linear operator $\Delta_{\text{BV}}: \overline{H}^n(\pi_{\text{BV}}) \rightarrow \overline{H}^n(\pi_{\text{BV}})$ is a differential,*

$$(\Delta_{\text{BV}})^2: \overline{H}^n(\pi_{\text{BV}}) \rightarrow 0.$$

Proof. Using equation (6) twice, we may write a repeated application of Δ_{BV} on $\mathcal{H} = \int h(x, [q], q^\dagger) d^n x \in \overline{H}^n(\pi_{\text{BV}})$ as

$$\Delta_{\text{BV}}^2(\mathcal{H}) = \int \frac{\overleftarrow{\delta}}{\delta q^\alpha} \frac{\overleftarrow{\delta}}{\delta q_\alpha^\dagger} \frac{\overleftarrow{\delta}}{\delta q^\beta} \frac{\overleftarrow{\delta}}{\delta q_\beta^\dagger} \frac{\overleftarrow{\delta} h}{\delta q_\beta^\dagger} d^n x. \quad (8)$$

Consider the rightmost variational derivative,

$$\frac{\overleftarrow{\delta}}{\delta q_\beta^\dagger} = \frac{\overrightarrow{\partial}}{\partial q_\beta^\dagger} + \sum_{\tau \neq \emptyset} D_\tau \circ \frac{\overrightarrow{\partial}}{\partial q_{\beta, \tau}^\dagger}.$$

To the immediate left of this in equation (8) stands the next variational derivative, $\overleftarrow{\delta}/\delta q^\beta$. Since $\frac{\overleftarrow{\delta}}{\delta q^\beta} \circ D_\tau = 0$ for any β and τ , all of the terms in the sum above disappear, and the rightmost variational derivative $\overleftarrow{\delta}/\delta q_\beta^\dagger$ becomes just $\overrightarrow{\partial}/\partial q_\beta^\dagger$ in equation (8). A similar process happens with the second and the third variational derivatives. As to the last, leftmost one, all terms of this variational derivative that contain total derivatives do not contribute to the functional, because the integral over a total derivative is zero according to our convention on no boundary terms. Summarizing, we have

$$\Delta_{\text{BV}}(\mathcal{H}) = \int \frac{\overrightarrow{\partial}}{\partial q^\alpha} \frac{\overrightarrow{\partial}}{\partial q_\alpha^\dagger} \frac{\overrightarrow{\partial}}{\partial q^\beta} \frac{\overrightarrow{\partial} h}{\partial q_\beta^\dagger} d^n x.$$

Since the middle two partial derivatives in this expression commute, this becomes

$$= \int \frac{\overrightarrow{\partial}}{\partial q^\alpha} \frac{\overrightarrow{\partial}}{\partial q^\beta} \frac{\overrightarrow{\partial}}{\partial q_\alpha^\dagger} \frac{\overrightarrow{\partial}}{\partial q_\beta^\dagger} (h) d^n x.$$

This is the composition of an expression which is symmetric in α and β (the left two partial derivatives) and an expression which is antisymmetric in α and β (the right two partial derivatives). Therefore it is zero. \blacksquare

As we did before with the Schouten bracket, we now extend the BV-Laplacian from the space $\overline{H}^n(\pi_{\text{BV}})$ of building blocks (1) to the linear subspace $\overline{\mathfrak{M}}^n(\pi_{\text{BV}})$ generated in $\mathfrak{M}^n(\pi_{\text{BV}}) = \text{Map}(\Gamma(\pi_{\text{BV}}) \rightarrow \mathbb{k})$ by elements from $\overline{H}^n(\pi_{\text{BV}})$ under multiplication in \mathbb{k} of their values at sections $s \in \Gamma(\pi_{\text{BV}})$.

Theorem 2. *The geometric construction of the BV-Laplacian in Definition 2 extends to the linear operator Δ_{BV} on the space $\overline{\mathfrak{M}}^n(\pi_{\text{BV}})$ such that*

$$\begin{aligned} \Delta_{\text{BV}}(F \cdot G)(s) &= (\Delta_{\text{BV}}(F) \cdot G)(s) + (-)^{\text{gh}(F)} \llbracket F, G \rrbracket(s) + (-)^{\text{gh}(F)} (F \cdot \Delta_{\text{BV}}(G))(s) \\ &= (\Delta_{\text{BV}} F)(s) \cdot G(s) + (-)^{\text{gh}(F)} \llbracket F, G \rrbracket(s) + (-)^{\text{gh}(F)} F(s) \cdot (\Delta_{\text{BV}} G)(s), \end{aligned} \quad (9)$$

for any $F, G \in \overline{\mathfrak{M}}^n(\pi_{\text{BV}})$ and all $s \in \Gamma(\pi_{\text{BV}})$; the antibracket $\llbracket \cdot, \cdot \rrbracket$ measures the deviation of Δ_{BV} from being a derivation.

Proof. Without loss of generality let us assume that the local functionals $F = F_1 \cdot \dots \cdot F_{i'}$ and $G = G_1 \cdot \dots \cdot G_{i''}$ are homogeneous terms in (1), consisting of i' and i'' building blocks from $\overline{H}^n(\pi_{\text{BV}})$, respectively. Following Remark 3, we then consider the product,

$$\underbrace{\pi_{\text{BV}} \times \dots \times \pi_{\text{BV}}}_{i' \text{ copies}} \times \underbrace{\pi_{\text{BV}} \times \dots \times \pi_{\text{BV}}}_{i'' \text{ copies}} = \pi_{\text{BV}}^{(1)} \times \dots \times \pi_{\text{BV}}^{(i')} \times \pi_{\text{BV}}^{(1)} \times \dots \times \pi_{\text{BV}}^{(i'')},$$

of $i' + i''$ copies of the BV-bundle π_{BV} . The total space of this product of bundles is endowed with the Cartan differential

$$d_{\mathcal{C}} = d'_{\mathcal{C}} + d''_{\mathcal{C}}$$

where $d'_{\mathcal{C}}$ is the Cartan differentials on the i' copies of π_{BV} of F , and $d''_{\mathcal{C}}$ is the Cartan differentials on the i'' copies of π_{BV} of G .

We start from the left hand side of formula (7):

$$\begin{aligned} \Delta_{\text{BV}}(F \cdot G) &= \left\langle \overleftarrow{\delta}_{\mathbf{q}}(\overleftarrow{\delta}_{\mathbf{q}^\dagger}(F \cdot G)) \right\rangle = \left\langle \left(\overleftarrow{\delta}'_{\mathbf{q}} + \overleftarrow{\delta}''_{\mathbf{q}} \right) \left(\overleftarrow{\delta}'_{\mathbf{q}^\dagger} + \overleftarrow{\delta}''_{\mathbf{q}^\dagger} \right) (F \cdot G) \right\rangle \\ &= \left\langle \left(\overleftarrow{\delta}'_{\mathbf{q}} + \overleftarrow{\delta}''_{\mathbf{q}} \right) \left(\overleftarrow{\delta}'_{\mathbf{q}^\dagger} F \cdot G + (-)^{\text{gh}(F)} F \cdot \overleftarrow{\delta}''_{\mathbf{q}^\dagger} G \right) \right\rangle \\ &= \left\langle \overleftarrow{\delta}'_{\mathbf{q}} \overleftarrow{\delta}'_{\mathbf{q}^\dagger} F \cdot G - \overleftarrow{\delta}'_{\mathbf{q}^\dagger} F \cdot \overleftarrow{\delta}''_{\mathbf{q}} G + (-)^{\text{gh}(F)} \overleftarrow{\delta}'_{\mathbf{q}} F \cdot \overleftarrow{\delta}''_{\mathbf{q}^\dagger} G + (-)^{\text{gh}(F)} F \cdot \overleftarrow{\delta}''_{\mathbf{q}} \overleftarrow{\delta}'_{\mathbf{q}^\dagger} G \right\rangle. \end{aligned}$$

The second term carries a minus sign because the anticommuting differentials $\overleftarrow{\delta}'_{\mathbf{q}^\dagger}$ and $\overleftarrow{\delta}''_{\mathbf{q}}$ have swapped positions in it. Reversing the direction of the arrow of $\overleftarrow{\delta}'_{\mathbf{q}^\dagger} F$ in this term (i.e., the odd-parity variation $\delta \mathbf{q}^\dagger$ is transported to the right instead of left), it gains an extra sign $(-)^{\text{gh}(F)-1}$. A similar arrow reversion $\overleftarrow{\delta}_{\mathbf{q}} \mapsto \overrightarrow{\delta}_{\mathbf{q}}$ in the third term does not produce a minus sign. Then the third and second terms combine to the Schouten bracket, while we recognize the first and the last one as the Laplacian acting on F and G , respectively:

$$\Delta_{\text{BV}}(F \cdot G) = \left\langle \overleftarrow{\delta}'_{\mathbf{q}} \overleftarrow{\delta}'_{\mathbf{q}^\dagger} F \cdot G + (-)^{\text{gh}(F)} \left(\overrightarrow{\delta}'_{\mathbf{q}} F \wedge \overleftarrow{\delta}''_{\mathbf{q}^\dagger} G + \overrightarrow{\delta}'_{\mathbf{q}^\dagger} F \wedge \overleftarrow{\delta}''_{\mathbf{q}} G \right) + (-)^{\text{gh}(F)} F \cdot \overleftarrow{\delta}''_{\mathbf{q}} \overleftarrow{\delta}'_{\mathbf{q}^\dagger} G \right\rangle.$$

Finally, we couple the Cartan differentials of the BV-variables and the assertion follows. \blacksquare

Contrary to the conventional BV-Laplacian (see Proposition 3 on page 23), the BV-Laplacian Δ_{BV} defined here does *not* always satisfy the equation $\Delta_{\text{BV}}(\llbracket F, G \rrbracket) = \llbracket \Delta_{\text{BV}} F, G \rrbracket + (-)^{\text{gh}(F)-1} \llbracket F, \Delta_{\text{BV}} G \rrbracket$. The reason is that if we were to proceed with the calculation of $\Delta_{\text{BV}}(\llbracket F, G \rrbracket)$ as in the proof of Proposition 3 (see below), it is no longer permissible to swap the symbols $\delta/\delta q^\alpha$ – which here stand for the *variational* derivative – with each other. Therefore the four remaining terms in equation (17) do not cancel. To see this explicitly, take the following counterexample.¹²

Counterexample 4. Let the base and fibre both be one-dimensional, and set

$$F = \int q^\dagger q q_{xx} \, dx \quad \text{and} \quad G = \int q_{xx}^\dagger \cos q \, dx.$$

Let f and g be the two integrands. We calculate

$$\begin{aligned} \frac{\overrightarrow{\delta f}}{\delta q} &= q^\dagger q_{xx} + D_x^2(q^\dagger q) = q^\dagger q_{xx} + q_{xx}^\dagger q + 2q_x^\dagger q_x + q^\dagger q_{xx} = 2q^\dagger q_{xx} + q_{xx}^\dagger q + 2q_x^\dagger q_x, \\ \frac{\overrightarrow{\delta f}}{\delta q^\dagger} &= q q_{xx}, \quad \frac{\overleftarrow{\delta g}}{\delta q} = -q_{xx}^\dagger \sin q, \\ \frac{\overleftarrow{\delta g}}{\delta q^\dagger} &= D_x^2(\cos q) = D_x(-q_x \sin q) = -q_{xx} \sin q - q_x^2 \cos q. \end{aligned} \quad (10)$$

(Note that since all four variational derivatives contain at most one parity-odd q^\dagger or its derivatives, the directions of the arrows do not actually matter – i.e., switching their direction does not result in minus signs.) Consider $\Delta_{\text{BV}}(\llbracket F, G \rrbracket)$. As was noted in Proposition 1, we may write $\Delta_{\text{BV}}(\mathcal{H}) = \int (\partial/\partial q) \circ (\partial/\partial q^\dagger)(\mathcal{H})$ for any $\mathcal{H} \in \overline{H}^n(\pi_{\text{BV}})$; therefore, only terms in which $\llbracket F, G \rrbracket$ carries at least one q^\dagger without derivatives with respect to the base space survive. This implies that in the first term of the bracket, $\overrightarrow{\delta f}/\delta q \cdot \overleftarrow{\delta g}/\delta q^\dagger$, the second and third term of the right hand side of (10) do not contribute, so we need not take them into account:

$$\llbracket F, G \rrbracket = \int (2q^\dagger q_{xx}(-q_{xx} \sin q - q_x^2 \cos q) + \cdots - q q_{xx} \cdot (-q_{xx}^\dagger \sin q)) \, dx,$$

where the dots indicate the omitted terms. For the same reason, the last term also does not contribute.

We calculate

$$\begin{aligned} \Delta_{\text{BV}}(\llbracket F, G \rrbracket) &= -2 \int \frac{\partial}{\partial q} \frac{\partial}{\partial q^\dagger} (q^\dagger q_{xx}^2 \sin q + q^\dagger q_{xx} q_x^2 \cos q) \, dx \\ &= -2 \int \frac{\partial}{\partial q} (q_{xx}^2 \sin q + q_{xx} q_x^2 \cos q) \, dx \\ &= -2 \int (q_{xx}^2 \cos q - q_{xx} q_x^2 \sin q) \, dx, \end{aligned}$$

whose integrand is not cohomologically trivial (as may be seen by calculating its variational derivative, which gives nonzero).

Now $\Delta_{\text{BV}} F = \int q_{xx} \, dx \cong 0$, so $\llbracket \Delta_{\text{BV}} F, G \rrbracket = 0$. On the other hand, g has no q^\dagger without any x -derivatives in it so $\Delta_{\text{BV}} G = 0$, so $\llbracket F, \Delta_{\text{BV}} G \rrbracket = 0$ as well. In conclusion,

$$\llbracket \Delta_{\text{BV}} F, G \rrbracket + (-)^{\text{gh}(F)-1} \llbracket F, \Delta_{\text{BV}} G \rrbracket = 0 \neq \Delta_{\text{BV}}(\llbracket F, G \rrbracket). \quad (11)$$

¹²We warn the reader that all claims in Counterexample 4 about any equalities between functionals (specifically, for $\Delta_{\text{BV}} F$ and $\Delta_{\text{BV}} G$ or for $\llbracket F, G \rrbracket$ and $\Delta(\llbracket F, G \rrbracket)$, etc.) should be viewed as the classical parable about a cage which contains an elephant but carries an inscription “BUFFALO” — one may not trust own eyes. We refer to Remark 6 in section 4 and also to Example 6 on p. 21 in which we explicitly calculate the objects ΔF and ΔG and $\Delta(\llbracket F, G \rrbracket)$, confirming the validity of equality (11).

This counterexample shows that we have reached the limits of a jet-space approach to the BV-geometry via Vinogradov's \mathcal{C} -spectral sequence $E_i^{p,q}$ (specifically, the upper line $E_1^{n,q}$ of its first term such that $E_1^{n,0} = \overline{H}^n(\pi_{\text{BV}})$). Let us note, however, that until now we operated with the objects $d_{\mathcal{C}}q^\alpha$ and δq^α or $d_{\mathcal{C}}q_\alpha^\dagger$ or δq_α^\dagger by viewing them as the Cartan differentials of dependent variables (essentially, the De Rham differentials).

We now pass from the space of infinite jets of sections in the BV-bundle π_{BV} to the space of sections $\Gamma(\pi \times_M \Pi\hat{\pi})$, that is, to the (anti)fields. Retaining the full arsenal of already known objects and structures, we let the approach become slightly more functional analytic. For example, we recall that both π and $\Pi\hat{\pi}$ are vector bundles over the base manifold M and that a section's *variation* $\delta s = (\delta s^\alpha, \delta s_\beta^\dagger)$ of $s \in \Gamma(\pi \times_M \Pi\hat{\pi})$ itself is a section of the bundle $T(\pi \times_M \Pi\hat{\pi})$.

4 The functional definition of BV-Laplacian

To approach the self-regularization in a genuine definition of the Batalin–Vilkovisky Laplacian, we analyze first the basic geometry of variation of functionals. Namely, let us study the interrelation of bundles in the course of integration by parts, the implications of the locality postulate, and a rigorous construction of the iterated functional derivatives.

The core of difficulties which became manifest in Counterexample 4 is that a use of the vector bundle π_{BV} in the construction of the space $\overline{\mathfrak{M}}(\pi_{\text{BV}})$, which contains formal sums of products $F_1 \dots F_k$ of integral functionals $F_i \in \overline{H}^n(\pi_{\text{BV}})$, is insufficient to grasp the full geometry of the calculus of variations. Indeed, several important identities combining the Schouten bracket with the BV-Laplacian do not hold; such identities involve higher-order variational derivatives, but those need to be regularized (or proclaimed permutable) whenever one inspects a response of a functional to a shift of its argument along different test sections, i. e., in several independent directions. To circumvent the difficulties, we shall enlarge the space of functionals in such a way that there is enough room to store the information about test shifts. Paradoxally, this generalization not only encodes properly the variations of (anti)fields but also communicates to the functionals a kind of “memory” of the way in which they were obtained from primary functionals (essentially, from the observables). The effect of the functionals' memory is manifest through the existence of synonymic mappings $\Gamma(\pi_{\text{BV}}) \rightarrow \mathbb{k}$ which yield equal numbers for every given section yet which belong to non-isomorphic spaces and behave differently under multiplication by using the Schouten bracket or under application of the BV-Laplacian.

This approach resolves the obstructions in a problem of intrinsic regularization of the Schouten bracket and the BV-Laplacian; the newly defined objects match in all standard ways (see Theorems 3 and 4 below).

For the sake of brevity, we denote by $[u]$ a differential dependence on the unknown variables (specifically to the BV-setup, a dependence on $[q]$ and $[q^\dagger]$).

Example 5. In particular, we shall explain why the conventional formula

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} F(s + \varepsilon \cdot \overleftarrow{\delta} s) = \int_M dx \, \delta s(x) \cdot \left. \frac{\overleftarrow{\delta} f(x, [u])}{\delta u} \right|_{j_x^\infty(s)} \quad (12)$$

for calculation of a functional's response to a test shift of its argument is a *consequence* of the definition — but not itself a definition. Containing a single functional derivative of $F = \int f(x, [u]) dx \in \overline{H}^n(\pi_{\text{BV}})$, formula (12) is especially instructive: we claim that a simplicity with which a correct expression is obtained by just one step is misleading; it hides a longer reasoning of which the right-hand side of (12) is an implication. Indeed, let us notice that the left-hand side of (12) refers to *three* bundles (namely, π_{BV} for the integral functional $F \in \overline{H}^n(\pi_{\text{BV}})$, the

vector bundle $\pi \times_M \Pi \widehat{\pi}$ for a background section s , and the tangent bundle $T(\pi \times_M \Pi \widehat{\pi})$ for the shift section δs .¹³ However, these domains of definition merge to the integration manifold M in the right-hand side. Therefore, from (12) it remains unclear whether the variational derivative $\overleftarrow{\delta} f / \delta \mathbf{u}$ refers to one (which would be false) or two (true!) copies of M . (In fact, we have that $\overleftarrow{\delta} f(x, [\mathbf{u}]) / \delta \mathbf{u} = \sum_{|\sigma| \geq 0} \left(-\frac{d}{dy}\right)^\sigma \Big|_{y=x} (\overrightarrow{\delta} f(x, [\mathbf{u}]) / \partial \mathbf{u}_\sigma)$.) We argue that the multiplication \cdot in the integrand of (12) is a result of the coupling in the defining equality

$$\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} F(s + \varepsilon \cdot \overleftarrow{\delta} s) = \int_M dx \int_M dy \left\langle \delta s(y), \left(-\frac{d}{dy}\right)^\sigma \left(\frac{\overrightarrow{\delta} f(x, [\mathbf{u}])}{\partial \mathbf{u}_\sigma} \right) \Big|_{j_x^\infty(s)} \right\rangle.$$

The mechanism of congruence $y = x$ and the geometric self-regularization of Dirac's delta-distribution, which is provided by that mechanism, appeals to the coupling

$$\int_M a(y) \Big|_{y=x} \cdot b(x) dx = \int_M dx \int_M dy \langle a(y), b(x) \rangle$$

of (co)vector fields $\langle a|$ and $|b\rangle$ on M . Thus, as a by-product we clarify the derivation of the Euler–Lagrange equation $\delta F(s_0) = 0$ upon the stationary points s_0 of an action functional F .

However, for identities containing several variational derivatives it becomes crucial that the differential of an integral functional's density is always referred to the “private” copy of the base $M \ni x$ owned in $\overline{H}^n(\pi_{\text{BV}})$ by that functional, whereas the integrations by parts are always performed with respect to the individual domains of definition $M \ni y_1, \dots, M \ni y_k$ of the variations. A restriction to the diagonal $x = y_1 = \dots = y_k$ stems from the coupling of those auxiliary vector bundles' sections $\delta s_1, \dots, \delta s_k$ with the k -th order variational derivative; the latter is a section of the dual vector bundle $T^*(\pi \times_M \Pi \widehat{\pi})$ whenever any $k - 1$ variations are viewed as parameters in the coupling.

Let us remember that the initial bundle π and the bundles π of fields and $\widehat{\pi}$ of antifields (such that $\overline{J}_\pi^\infty(\pi_{\text{BV}}) = J^\infty(\pi) \times_M J^\infty(\Pi \widehat{\pi})$) are *vector* bundles over M . This means that the fibre over each point $x \in M$ in $\pi \times_M \Pi \widehat{\pi}$ is a vector space $V_x \oplus \Pi V_x^\dagger$ endowed with a \mathbb{k} -linear structure; the sum of two sections $s \in \Gamma(\pi \times_M \Pi \widehat{\pi})$ and $\varepsilon \cdot \delta s \in \Gamma(T(\pi \times_M \Pi \widehat{\pi}))$ is defined pointwise on M : we have that $(s + \varepsilon \cdot \delta s)(x) = s(x) + \varepsilon \cdot \delta s(x) \in V_x \oplus \Pi V_x^\dagger$ (here we use the vector space structure of the fibre $V_x \oplus \Pi V_x^\dagger$, also identifying it with its own tangent space at $s(x)$ so that the sum of sections makes sense). This is the locality postulate which is brought into the model by hand. (Note that in what follows we avoid an identification of $\Gamma(T(\pi \times_M \Pi \widehat{\pi}))$ with $\Gamma(\pi \times_M \Pi \widehat{\pi})$ even if we deal with the fibrewise-constant shifts δs for which $\delta s(s_1(x)) = \delta s(s_2(x)) =: \delta s(x)$ for any $s_1, s_2 \in \Gamma(\pi \times_M \Pi \widehat{\pi})$ at $x \in M$.) The locality postulate is a mechanism which communicates the value of $\varepsilon \cdot \delta s$ at $y \in M$ or the values at $y := x$ of derivatives of $\varepsilon \cdot \delta s$ along its domain of definition to the geometry that carries the section s and handles its value at $x \in M$.

We now transform the postulate of locality (which stemmed from the initial hypothesis that the bundles' fibres are vector spaces) and the multiplication \cdot of the objects' coefficients to the mechanism of congruence $y = x$ of attachment points for (co)vector fields $\langle | \rangle(y)$ or $| \rangle(y)$ and $| \rangle(x)$ or $\langle | \rangle(x)$ in the coupling $\langle, \rangle: T_{s(y)} \Pi V_y^\dagger \times T_{s(x)} V_x \rightarrow \mathbb{k}$ and $\langle, \rangle: T_{s(y)} V_y \times T_{s(x)} \Pi V_x^\dagger \rightarrow \mathbb{k}$ (respectively, we have $\langle, \rangle: T_{s(y)}^* V_y \times T_{s(x)} V_x \rightarrow \mathbb{k}$ and $\langle, \rangle: T_{s(y)} V_y \times T_{s(x)}^* V_x \rightarrow \mathbb{k}$ up to the

¹³Because the fibre $V_x \oplus \Pi V_x^\dagger$ over $x \in M$ in the bundle $\pi \times_M \Pi \widehat{\pi}$ is a vector space, its tangent space at a point $s(x)$ is isomorphic to the fibre. Still we emphasize that there is a distinction between s and δs (they are sections of different bundles). We note further that $\delta s \in \Gamma(T(\pi \times_M \Pi \widehat{\pi}))$ is postulated to be independent of $s \in \Gamma(\pi \times_M \Pi \widehat{\pi})$, i.e., the shift of s at x is the same at all values $s(x) \in V_x \oplus \Pi V_x^\dagger$ so that the notation $\delta s(x)$ makes sense (a rigorous formula would be $\delta s(s(x)) \in T_{s(x)}(V_x \oplus \Pi V_x^\dagger)$). The independence of test shifts of the background field manifests the translation invariance of the measure in Feynman's path integral (see section 5).

introduction of ghost parity). This will be the key element in a self-regularization procedure for $[\![\ , \]\!]$ and Δ_{BV} when, at the second step of the reasoning, we explicitly use the duality between the halves of sections for $\pi \times_M \Pi \hat{\pi}$. The operational definitions of Schouten bracket and BV-Laplacian amount to surgery algorithms for reconfigurations of such couplings (see Definition 3 below). Currently, we deal with iterated functional derivatives in full generality, not actually referring to the composite structure of $\pi \times_M \Pi \hat{\pi}$ and of the associated tangent bundle.

Let us extend the intrinsic geometry of each building block in construction (1) of the composite functionals space $\overline{\mathfrak{M}}^n(\pi_{\text{BV}})$. Leaving the multiplication of integral functionals intact, we replace the bundle π_{BV} for $j^\infty(s)$ in each $\overline{H}^n(\pi_{\text{BV}})$ by the product of the BV-bundle π_{BV} with k copies $T\pi_{\text{BV}} \times \dots \times T\pi_{\text{BV}}$ of its tangent bundle; the future sections of such product bundle will be composed by $s \in \Gamma(\pi \times_M \Pi \hat{\pi})$ and k variations $\delta s_1, \dots, \delta s_k$ which are constant along each fibre in π_{BV} (here $k \geq 0$). We introduce the space¹⁴

$$\begin{aligned} \overline{\mathfrak{N}}^n(\pi_{\text{BV}}) &= \bigoplus_{i=1}^{+\infty} \bigotimes_{\mathbb{k}}^i \bigoplus_{k=0}^{+\infty} \overline{H}^{n(1+k)}(\pi_{\text{BV}} \times \underbrace{T\pi_{\text{BV}} \times \dots \times T\pi_{\text{BV}}}_{k \text{ copies}} \times \underbrace{\phantom{T\pi_{\text{BV}} \times \dots \times T\pi_{\text{BV}}}}_{k \text{ variations}}) \subseteq \\ &\subseteq \text{Map}(\Gamma(\pi_{\text{BV}}) \times \Gamma(T\pi_{\text{BV}}) \times \dots \times \Gamma(T\pi_{\text{BV}}) \rightarrow \mathbb{k}). \end{aligned}$$

Note that the product of bundles in the above formula is not a Whitney sum over the common base M ; conversely, each variation δs_j brings its own copy of the base into the geometry. An analogous technique was mentioned in Remark 3 on p. 7; that approach will be used again – at a later stage – in the functional definition of the Schouten bracket that combines two functionals from $\overline{\mathfrak{N}}^n(\pi_{\text{BV}}) \supsetneq \overline{\mathfrak{M}}^n(\pi_{\text{BV}})$ to one functional. However, we now analyze the mechanism of merging the bases $M \times M \times \dots \times M$ in the product of bundles inside each factor $F_j \in \overline{H}^{n(1+k)}(\pi_{\text{BV}} \times T\pi_{\text{BV}} \times \dots \times T\pi_{\text{BV}})$ of a homogeneous term $F_1 \dots F_i$ of $\overline{\mathfrak{N}}^n(\pi_{\text{BV}})$.

Consider an integral functional $F = \int f(x, [\mathbf{u}]) dx \in \overline{H}^n(\pi_{\text{BV}}) \subset \overline{\mathfrak{M}}^n(\pi_{\text{BV}})$ and let $s \in \Gamma(\pi \times_M \Pi \hat{\pi})$. To inspect a response of the functional $F: s \mapsto F(s) \in \mathbb{k}$ to an infinitesimal shift of its argument s by k test sections $\delta s_1, \dots, \delta s_k$ of the vector bundle $T(\pi \times_M \Pi \hat{\pi})$, we take

$$\left. \frac{d}{d\varepsilon'} \right|_{\varepsilon'=0} \left. \frac{d}{d\varepsilon''} \right|_{\varepsilon''=0} \dots \left. \frac{d}{d\varepsilon^{(k)}} \right|_{\varepsilon^{(k)}=0} F(s + \varepsilon' \cdot \overleftarrow{\delta s_1} + \dots + \varepsilon^{(k)} \cdot \overleftarrow{\delta s_k}).$$

By using the chain rule we have that

$$\begin{aligned} &\left. \frac{d}{d\varepsilon'} \right|_{\varepsilon'=0} \dots \left. \frac{d}{d\varepsilon^{(k)}} \right|_{\varepsilon^{(k)}=0} \int_M dx f(x, [\mathbf{u}]) \Big|_{j_x^\infty(s) + \varepsilon^{(i)} \cdot j_{y_i}^\infty(\overleftarrow{\delta s_i})} \Big|_{y_i:=x} = \\ &= \int_M dx \left\{ \sum_{j=1}^k \sum_{\substack{|\sigma_j| \geq 0, \\ i_j=1, \dots, \text{rank}(\pi \times_M \Pi \hat{\pi})}} \left(\frac{\partial}{\partial y_1} \right)^{\sigma_1} \Big|_{y_1} (\delta s_1^{i_1}) \dots \left(\frac{\partial}{\partial y_k} \right)^{\sigma_k} \Big|_{y_k} (\delta s_k^{i_k}) \cdot \right. \\ &\quad \left. \cdot \frac{\vec{\partial}^k f(x, [\mathbf{u}])}{\partial u_{\sigma_1}^{i_1} \dots \partial u_{\sigma_k}^{i_k}} \Big|_{j_x^\infty(s)} \right\} \Big|_{y_j:=x}. \end{aligned}$$

We emphasize that the derivatives of a density f for F refer to the base $M \ni x$ of that integral functional. At the same time, the derivatives of fibrewise-constant test shifts δs_j are taken with respect to their own domains of definition $M \ni y_j$ so that it is the numbers from \mathbb{k} which are communicated to the operation of multiplication \cdot at the (very last) moment of restriction to the diagonal $y_j = x$.

¹⁴By construction, elements of $\overline{\mathfrak{N}}^n(\pi_{\text{BV}})$ will determine linear maps with respect to each variation $\varepsilon_j \cdot \delta s_j(y_j)$ in the limit $\varepsilon_j \rightarrow 0$.

To substantiate this mechanism, let us deal not with the *coefficients* $\delta s_j^{ij}(y_j)$ of sections δs_j and not with the multiplication \cdot in \mathbb{k} , but let us reformulate the picture in terms of (co)frame fields over M and their couplings. Recall the standard geometry of dual bases located in the fibres of a Whitney sum of two vector bundles over points $x, y \in M$ of their base: if $x = y$, the coupling of two representatives in the fibres works out an appropriate number from \mathbb{k} ; otherwise, if $x \neq y$, the coupling is automatically zero.

Note that at each $y_i \in M$ the value $(\partial/\partial y_i)^{\sigma_i}(\delta s_i)(y_i)$ is an element of the vector space $T_{s(y_i)}(V_{y_i} \oplus \Pi V_{y_i}^\dagger)$ and the value $\vec{\partial}f(x, [\mathbf{u}])/\partial \mathbf{u}_\sigma|_{j_x^\infty(s)}$ at $x \in M$ belongs to the dual vector space $T_{s(x)}^*(V_x \oplus \Pi V_x^\dagger)$ over x . We repeat that the derivatives $(\partial/\partial y_i)^{\sigma_i}$ along the base manifold $M \ni y_i$ act on the coefficients of sections but not on the frame fields. This implies that the linear part of a response of F to a shift of its argument away from s is equal to

$$\begin{aligned} \frac{d}{d\varepsilon'} \Big|_{\varepsilon'=0} \cdots \frac{d}{d\varepsilon^{(k)}} \Big|_{\varepsilon^{(k)}=0} F(s + \varepsilon' \cdot \overleftarrow{\delta s_1} + \dots + \varepsilon^{(k)} \cdot \overleftarrow{\delta s_k}) = \\ = \int_M dx \int_M dy_1 \cdots \int_M dy_k \left\langle \begin{array}{cc} (\partial/\partial y_1)^{\sigma_1}(\delta s_1)(y_1), & \vec{\partial}/\partial \mathbf{u}_{\sigma_1} \\ \vdots & \vdots \\ (\partial/\partial y_k)^{\sigma_k}(\delta s_k)(y_k), & \vec{\partial}/\partial \mathbf{u}_{\sigma_k} \end{array} (f(x, [\mathbf{u}])) \Big|_{j_x^\infty(s)} \right\rangle. \end{aligned}$$

Each line within \langle, \rangle contains a coupling of the dual bases; it forces the points y_i and x to coincide at the moment of evaluation at $(1+k)$ sections (but not earlier). Integrating by parts, we obtain (by using the agreement on the absence of boundary terms)

$$\begin{aligned} \cong \int_M dx \int_M dy_1 \cdots \int_M dy_k \\ \left\langle \delta s_1(y_1) \dots \delta s_k(y_k), \left(-\frac{\partial}{\partial y_1} \right)^{\sigma_1} \Big|_{y_1} \cdots \left(-\frac{\partial}{\partial y_k} \right)^{\sigma_k} \Big|_{y_k} \left(\frac{\vec{\partial}^k f(x, [\mathbf{u}])}{\partial \mathbf{u}_{\sigma_1} \dots \partial \mathbf{u}_{\sigma_k}} \right) \Big|_{j_x^\infty(s)} \right\rangle. \end{aligned} \quad (13)$$

Clearly, before its evaluation at a concrete section $s \in \Gamma(\pi \times_M \Pi \widehat{\pi})$ and at given shifts $\delta s_1, \dots, \delta s_k$, the object we are dealing with exists as an element of top-cohomology group $\overline{H}^{n(1+k)}(\pi_{\text{BV}} \times T\pi_{\text{BV}} \times \dots \times T\pi_{\text{BV}})$. In particular, various identities involving variational derivatives (e.g., see (16) below) are equalities between the maps taking s and all its variations δs_i to a number from \mathbb{k} . In conclusion, all the couplings which force the congruence of points y_i and $x \in M$ are never performed in the course of verification of such identities within $\overline{\mathfrak{N}}^n(\pi_{\text{BV}})$. Moreover, whenever an extra derivative, corresponding to $\delta s_{k+1}(y_{k+1})$, falls on an object whose intrinsic geometry already encodes k shifts of s , the $(k+1)$ -th order derivative of the initial functional's density f is still referred to its own base $M \ni x$; it does not feel the presence of any test shifts. Consequently, each functional's derivative adds a new copy of its domain of definition and inserts an extra tangent bundle $T\pi_{\text{BV}}$ to the product $\pi_{\text{BV}} \times T\pi_{\text{BV}} \times \dots \times T\pi_{\text{BV}}$.

Remark 5. A one-step reduction to the diagonal $y = x$ has been illustrated in Example 5 on p. 15. Another case – with multiple functional derivatives – will be considered in Example 6 in order to show an instance of the surgery technique for reconfigurations of the couplings; the geometry of that model is specific to the BV-setup of (anti)fields.

Remark 6. There exist integral functionals $F \in \overline{H}^n(\pi_{\text{BV}})$ and $G \in \overline{H}^{n+2n}(\pi_{\text{BV}} \times \pi_{\text{BV}} \times \pi_{\text{BV}})$ and there are (specially chosen, see below) test shifts δs_1 and δs_2 such that $F(s) = G(s, \delta s_1, \delta s_2) \in \mathbb{k}$ for every $s \in \Gamma(\pi \times_M \Pi \widehat{\pi})$ but such that F and G behave differently with respect to the Schouten bracket or BV-Laplacian. For example, one can find $H \in \overline{H}^n(\pi_{\text{BV}})$ such that $\llbracket F, H \rrbracket \neq \llbracket G, H \rrbracket$ as elements of $\overline{\mathfrak{N}}^n(\pi_{\text{BV}})$ and $\llbracket F, H \rrbracket(s) \neq \llbracket G, H \rrbracket(s, \delta s_1, \delta s_2) \in \mathbb{k}$ or such that $\Delta_{\text{BV}}(F) \neq \Delta_{\text{BV}}(G)$ and $(\Delta_{\text{BV}}F)(s) \neq (\Delta_{\text{BV}}G)(s, \delta s_1, \delta s_2)$.

Namely, let $\delta s_1 = (\delta s_1^1, \dots, \delta s_1^N; 0, \dots, 0) \in \Gamma(T(\pi \times_M \Pi\hat{\pi}))$, here $N = \text{rank } \pi$, and let $\delta s_2 = (0, \dots, 0; \delta s_{2,1}^\dagger, \dots, \delta s_{2,N}^\dagger)$ such that $\delta s_1^i \cdot \delta s_{2,i}^\dagger = 1$ for every i (a possibility to have that normalization independent from a choice of local coordinates is guaranteed by the composite geometry of fibres in the vector bundle $\pi \times_M \Pi\hat{\pi}$). Let $f(x, [\mathbf{q}], [\mathbf{q}^\dagger])$ be a differential function on the jet space. Then the functionals

$$F := \int dx \sum_{i=1}^N \left\{ \delta s_1^i(x) \cdot \delta s_{2,i}^\dagger(x) \cdot \frac{\overleftarrow{\partial}^2 f(x, [\mathbf{q}], [\mathbf{q}^\dagger])}{\delta \mathbf{q}^i \delta \mathbf{q}_i^\dagger} \right\}$$

and (see (13) above

$$G := \int dx \int dy \int dz \left\{ \sum_{i=1}^N \sum_{|\sigma_1|, |\sigma_2| \geq 0} \delta s_1^i(y) \cdot \delta s_{2,i}^\dagger(z) \cdot \left(-\frac{d}{dy} \right)^{\sigma_1} \Big|_y \left(-\frac{d}{dz} \right)^{\sigma_2} \Big|_z \frac{\overrightarrow{\partial}^2 f(x, [\mathbf{q}], [\mathbf{q}^\dagger])}{\partial \mathbf{q}_{\sigma_1}^i \partial \mathbf{q}_{i, \sigma_2}^\dagger} \right\} \cdot \delta(y-x) \delta(z-x)$$

are indistinguishable as mappings to \mathbb{k} for every $s \in \Gamma(\pi \times_M \Pi\hat{\pi})$, yet the difference in the behaviour of F and G under basic operations of differential calculus provides a resolution to the obstructions in Counterexample 4, c. f. Example 6. We see that unequal elements from $\overline{\mathfrak{M}}^n(\pi_{\text{BV}})$ and $\overline{\mathfrak{N}}^n(\pi_{\text{BV}})$ can determine synonymic maps of the space of sections. At the same time, elements $F \in \overline{\mathfrak{M}}^n(\pi_{\text{BV}})$ could be viewed as primary functionals; their descendants $G \in \overline{\mathfrak{N}}^n(\pi_{\text{BV}})$ retain a kind of memory of the way in which they have been derived from some primary objects in $\overline{H}^n(\pi_{\text{BV}}) \subseteq \overline{\mathfrak{M}}^n(\pi_{\text{BV}})$. Moreover, such memory governs the behaviour of descendants in the course of analytic operations (see Example 6 below for a nontrivial synonym ΔG of the zero functional).

The above reasoning was unspecific to the BV-geometry; with elementary modifications in notation it could be applied, e.g., to a derivation of the Euler–Lagrange equations upon sections of a given vector bundle (see Example 5). We now use the initial assumption that the BV-bundle $\pi \times_M \Pi\hat{\pi}$ has a very special structure of its fibres. Namely, the dependent variables \mathbf{q} and \mathbf{q}^\dagger are grouped in two halves, each set's cardinality being equal to $m_0 + m_1 + \dots + m_\lambda =: N$. The basic vectors $\vec{e}^{\dagger,1}, \dots, \vec{e}^{\dagger,N}$ in the second summand of the fibre $V_x \oplus \Pi V_x^\dagger \simeq T_{s(x)}(V_x \oplus \Pi V_x^\dagger)$ over each $x \in M$ are dual to the respective vectors $\vec{e}_1, \dots, \vec{e}_N$, spanning V_x , under the \mathbb{k} -valued coupling $\langle \vec{e}_\alpha(x), \vec{e}^{\dagger,\beta}(y) \rangle = \delta_\alpha^\beta \cdot \delta(x-y)$. (Note that the coupling changes the ghost parity by convention; the vectors $\vec{e}^{\dagger,1}, \dots, \vec{e}^{\dagger,N}$ themselves carry odd parity by the construction of $\Pi\hat{\pi}$, but the Kronecker delta in the coupling is postulated to be ghost-parity even.) For any (smooth) choice of a frame field of basic vectors \vec{e}_α and $\vec{e}^{\dagger,\beta}$ over M , a local section $s \in \Gamma(\pi \times_M \Pi\hat{\pi})$ of the vector bundle at hand is then

$$s = s^1 \cdot \vec{e}_1 + \dots + s^N \cdot \vec{e}_N + s_1^\dagger \cdot \vec{e}^{\dagger,1} + \dots + s_N^\dagger \cdot \vec{e}^{\dagger,N},$$

for some \mathbb{k} -valued functions s^α and s_β^\dagger on M ; their smoothness class is governed by an external agreement which is brought in by hand.

Continuing this line of reasoning, we see that the test shifts $\delta s \in \Gamma(T(\pi \times_M \Pi\hat{\pi}))$ of the functionals' arguments inherit this splitting: $\delta s = (\delta s^\alpha, \delta s_\beta^\dagger)$. Furthermore, one could either employ the shifts $\delta s_1 = (\delta s^\alpha, 0)$ or $\delta s_2 = (0, \delta s_\beta^\dagger)$ of pure ghost parities (+) and (−), respectively. Alternatively, one can split a given section $\delta s = (\delta s^\alpha, \delta s_\beta^\dagger) = (\delta s^\alpha, 0) + (0, \delta s_\beta^\dagger)$ and then use these two terms at different stages. Let us point out a convenient normalization $\delta s^\alpha(x) \cdot \delta s_\alpha^\dagger(x) \equiv 1$

at all $x \in M$ and a fixed index $\alpha = 1, \dots, N$; this constraint is coordinate-independent owing to the duality of V_x and V_x^\dagger . (The right hand side of the identity is the unit that could bear the unconventional odd ghost parity; we let it be even.)

We introduce the shorthand notation

$$\left. \frac{\overleftarrow{\delta} f(x, [\mathbf{q}], [\mathbf{q}^\dagger])}{\delta q^\alpha(y)} \right|_{j_x^\infty(s)} = \sum_{|\sigma| \geq 0} \left(-\frac{\partial}{\partial y} \right)^\sigma \left(\frac{\overrightarrow{\delta} f(x, [\mathbf{q}], [\mathbf{q}^\dagger])}{\partial q_\sigma^\alpha} \right) \Big|_{j_x^\infty(s)}$$

and

$$\left. \frac{\overleftarrow{\delta} f(x, [\mathbf{q}], [\mathbf{q}^\dagger])}{\delta q_\beta^\dagger(y)} \right|_{j_x^\infty(s)} = \sum_{|\sigma| \geq 0} \left(-\frac{\partial}{\partial y} \right)^\sigma \left(\frac{\overrightarrow{\delta} f(x, [\mathbf{q}], [\mathbf{q}^\dagger])}{\partial q_{\beta, \sigma}^\dagger} \right) \Big|_{j_x^\infty(s)}$$

for the components of pure ghost parity functional derivatives. At every point of the base M and for a given section $s \in \Gamma(\pi \times_M \Pi \hat{\pi})$ and a functional $F \in \overline{\mathfrak{M}}^n(\pi_{\text{BV}})$, we have that

$$\left. \frac{\overleftarrow{\delta} f(x, [\mathbf{q}], [\mathbf{q}^\dagger])}{\delta q^\alpha(y)} \right|_{j_x^\infty(s)} \in T_{s(x)}^*(\Pi) V_x \quad \text{and} \quad \left. \frac{\overleftarrow{\delta} f(x, [\mathbf{q}], [\mathbf{q}^\dagger])}{\delta q_\beta^\dagger(y)} \right|_{j_x^\infty(s)} \in T_{s(x)}^*(\Pi) V_x^\dagger,$$

forcing the points y and x to coincide; the optional presence of the parity reversion operator indicates a possibly odd ghost parity $\text{gh}(F)$ of the functional itself.

Definition 3. For a fixed choice of the test shift $\delta s = (\delta s^\alpha, \delta s_\beta^\dagger) = (\delta s^\alpha, 0) + (0, \delta s_\beta^\dagger)$, the *functional BV-Laplacian* is the linear operator $\Delta: \overline{\mathfrak{M}}^n(\pi_{\text{BV}}) \rightarrow \overline{\mathfrak{M}}^n(\pi_{\text{BV}})$ defined for all $F \in \overline{\mathfrak{M}}^n(\pi_{\text{BV}})$ and $s \in \Gamma(\pi \times_M \Pi \hat{\pi})$ by the rule

$$(\Delta F)(s) = \left\langle \frac{d}{d\varepsilon'} \Big|_{\varepsilon'=0} \frac{d}{d\varepsilon''} \Big|_{\varepsilon''=0} F(s + \varepsilon' \overleftarrow{\delta} s^\alpha \cdot \vec{e}_\alpha + \varepsilon'' \overleftarrow{\delta} s_\beta^\dagger \cdot \vec{e}^{\dagger, \beta}) \right\rangle,$$

where the coupling goes as follows. By taking

$$\int_M dx \int_M dy_1 \int_M dy_2 \left\langle \delta s^\alpha(y_1) \delta s_\beta^\dagger(y_2), \frac{\overleftarrow{\delta^2} f(x, [\mathbf{q}], [\mathbf{q}^\dagger])}{\delta q^\alpha(y_1) \delta q_\beta^\dagger(y_2)} \Big|_{j_x^\infty(s)} \right\rangle,$$

we obtain an integrand with the couplings $\langle \cdot, \cdot \rangle: T_{s(y_1)} V_{y_1} \times T_{s(x)}^*(\Pi) V_x \rightarrow \mathbb{k}$ and $\langle \cdot, \cdot \rangle: T_{s(y_2)} \Pi V_{y_2}^\dagger \times T_{s(x)}^* \Pi V_x^\dagger \rightarrow \mathbb{k}$. Let these couplings be reattached to $\langle \cdot, \cdot \rangle: T_{s(y_1)} V_{y_1} \times T_{s(y_2)} \Pi V_{y_2}^\dagger \rightarrow \mathbb{k}$ and $\langle \cdot, \cdot \rangle: T_{s(x)}^* \Pi V_x \times T_{s(x)}^* \Pi V_x^\dagger \rightarrow \mathbb{k}$; in fact, this surgery algorithm is the operational definition. The coupling between dual objects $\vec{e}_\alpha(y)$ and $\vec{e}^{\dagger, \beta}(x)$ yields non-zero only if $y = x$, which makes the procedure possible and we have

$$(\Delta F)(s) = \sum_{\alpha, \beta=1}^N \iiint_M dx dy_1 dy_2 \delta s^\alpha(y_1) \delta s_\beta^\dagger(y_2) \cdot \frac{\overleftarrow{\delta^2} f(x, [\mathbf{q}], [\mathbf{q}^\dagger])}{\delta q^\alpha(y_1) \delta q_\beta^\dagger(y_2)} \Big|_{j_x^\infty(s)} \cdot \delta(y_1 - x) \cdot (y_2 - x),$$

which results in the conventional formula with a summation over the diagonal in $V \oplus \Pi V^\dagger$:

$$= \sum_{\alpha=1}^N \int_M dx \delta s^\alpha(x) \delta s_\alpha^\dagger(x) \frac{\overleftarrow{\delta^2} f(x, [\mathbf{q}], [\mathbf{q}^\dagger])}{\delta q^\alpha \delta q_\alpha^\dagger} \Big|_{j_x^\infty(s)}.$$

In particular, if $\delta s^\alpha(x) \delta s_\alpha^\dagger(x) \equiv 1 \in \mathbb{k}$ (or $\equiv 1^{(\Pi)} \in \Pi \mathbb{k}$, which is unusual¹⁵), then we say that Δ is the *normalized* functional BV-Laplacian.

In what follows we shall use the normalized Laplacian by default.

¹⁵The choice of $1^{(\Pi)} = \Pi(1)$ in the coupling preserves the ghost parity of the functional F even though the second functional derivative in the definition of $\Delta(F)$ does not.

Remark 7. The coupling of canonically dual components of a section δs is performed *last* in this operational definition. Until that moment the derivatives $\frac{d}{d\varepsilon'} \longleftrightarrow \overleftarrow{\delta}/\delta q^\alpha(y_1)$ and $\frac{d}{d\varepsilon''} \longleftrightarrow \overleftarrow{\delta}/\delta q^\dagger_\beta(y_2)$ refer to different copies of the base M and thus they can be freely swapped.

The normalized functional BV-Laplacian Δ is a constituent part of a broader technique which appeals to the use of test shifts $\delta s \in \Gamma(T(\pi \times_M \Pi \widehat{\pi}))$ and functional derivatives $\frac{d}{d\varepsilon}|_{\varepsilon=0} F(s+\varepsilon \cdot \delta s)$ for $F \in \overline{\mathfrak{N}}^n(\pi_{\text{BV}})$ instead of a use of Cartan differentials δq or δq^\dagger and of the variations $\overleftarrow{\delta} q F$ or $\overleftarrow{\delta} q^\dagger F$, respectively. The normalized operator Δ retains the properties of Δ_{BV} on jet spaces: namely, its linearity over \mathbb{k} on $\overline{\mathfrak{N}}^n(\pi_{\text{BV}})$, the product rule formula (9), and Proposition 1. Yet – by a transition to the spaces of sections $\Gamma(\pi \times_M \Pi \widehat{\pi})$ and $\Gamma(T(\pi \times_M \Pi \widehat{\pi}))$ – we gain a greater flexibility in our functional approach to the variations. Indeed, the (graded) permutability of functional derivatives is the key point: it resolves the obstructions which were illustrated by Counterexample 4.

From now on, we employ functional derivatives in the construction of the BV-Laplacian Δ and the Schouten bracket $\llbracket \cdot, \cdot \rrbracket$, for which the algorithm of reattachments and the delta function mechanisms were de-facto pronounced in section 2. The bracket $\llbracket \cdot, \cdot \rrbracket$ thus also retains all its properties. On top of that, we postulate the conventional normalization $\delta s^\alpha(x) \delta s^\dagger_\alpha(x) \equiv 1 \in \mathbb{k}$ at each α and each $x \in M$, so that the resulting expressions contain neither parity-swappings nor any traces of the test-shift coefficients of the couplings $\langle \vec{e}_\alpha(x), \vec{e}^{\dagger, \beta}(y) \rangle = \delta_\alpha^\beta \cdot \delta(x - y)$ for the frames in the fibres of the BV-bundle.

Example 6. Consider the integral functionals $F = \int q^\dagger q q_{x_1 x_1} dx_1$ and $G = \int q^\dagger_{x_2 x_2} \cos q dx_2$, c.f. Counterexample 4 on p. 14. Let us show that equality $\Delta(\llbracket F, G \rrbracket) = \llbracket \Delta F, G \rrbracket + \llbracket F, \Delta G \rrbracket$ *does hold* for these functionals whenever one uses the operational definitions of the Schouten bracket and Batalin–Vilkovisky Laplacian, hence making a proper distinction between the bases of bundles in the course of integrations by parts. (For the sake of brevity, we do not indicate the base points' congruences which occur due to the couplings. Still we explicitly refer the functionals' densities to their own domains with coordinates x_1 and x_2 .)

We have

$$\begin{aligned} \llbracket F, G \rrbracket &= \iiint \int dx_1 dx_2 dy_1 dy_2 \left\langle \underbrace{\left(q^\dagger q_{xx} + \frac{d^2}{dy_1^2} (q^\dagger q) \right)}_{x_1} \cdot \underbrace{\langle \delta q(y_1), \delta q^\dagger(y_2) \rangle}_{+1} \cdot \underbrace{\frac{d^2}{dy_2^2} (\cos q)}_{x_2} \right\rangle \\ &\quad + \iiint \int dx_1 dx_2 dy_1 dy_2 \left\langle \underbrace{(q q_{xx})}_{x_1} \cdot \underbrace{\langle \delta q^\dagger(y_1), \delta q(y_2) \rangle}_{-1} \cdot \underbrace{(-q^\dagger_{xx} \sin q)}_{x_2} \right\rangle. \end{aligned}$$

Therefore, one side of the expected equality is

$$\begin{aligned} \Delta(\llbracket F, G \rrbracket) &= \int dz_1 \int dz_2 \int dx_1 \int dx_2 \int dy_1 \int dy_2 \underbrace{\langle \delta q(z_1), \delta q^\dagger(z_2) \rangle}_{+1} \cdot \underbrace{\langle \delta q(y_1), \delta q^\dagger(y_2) \rangle}_{+1} \\ &\quad \cdot \left\langle \underbrace{\frac{d^2}{dz_1^2} (1)}_{x_1} \cdot \underbrace{\frac{d^2}{dy_2^2} (\cos q)}_{x_2} + \underbrace{q_{xx}}_{x_1} \cdot \underbrace{\frac{d}{dy_2^2} (-\sin q)}_{x_2} + \underbrace{\frac{d^2}{dy_1^2} (1)}_{x_1} \cdot \underbrace{\frac{d^2}{dy_2^2} (\cos q)}_{x_2} + \underbrace{\frac{d^2}{dy_1^2} (q)}_{x_1} \cdot \underbrace{\frac{d^2}{dy_2^2} (-\sin q)}_{x_2} \right\rangle \\ &\quad + \int dz_1 \int dz_2 \int dx_1 \int dx_2 \int dy_1 \int dy_2 \underbrace{\langle \delta q(z_1), \delta q^\dagger(z_2) \rangle}_{+1} \\ &\quad \cdot \left\langle \underbrace{q_{xx}}_{x_1} \cdot \underbrace{\frac{d^2}{dz_2^2} (-\sin q)}_{x_2} + \underbrace{\frac{d^2}{dz_1^2} (q)}_{x_1} \cdot \underbrace{\frac{d^2}{dz_2^2} (-\sin q)}_{x_2} + \underbrace{(q q_{xx})}_{x_1} \cdot \underbrace{\frac{d^2}{dz_2^2} (-\cos q)}_{x_2} \right\rangle \cdot \underbrace{\langle \delta q^\dagger(y_1), \delta q(y_2) \rangle}_{-1}. \end{aligned}$$

The respective pairs of underlined terms cancel out and there remains only

$$= \int \cdots \int dz_1 dz_2 dx_1 dx_2 dy_1 dy_2 \underbrace{\langle \delta q(z_1), \delta q^\dagger(z_2) \rangle}_{+1} \cdot \underbrace{\langle (qq_{xx}) \cdot \frac{d^2}{dz_1^2} (-\cos q) \rangle}_{x_1} \cdot \underbrace{\langle \delta q^\dagger(y_1), \delta q(y_2) \rangle}_{-1}. \quad (14)$$

On the other hand, we obtain that

$$\Delta F = \iiint dz_1 dz_2 dx_1 \underbrace{\langle \delta q(z_1), \delta q^\dagger(z_2) \rangle}_{+1} \cdot \underbrace{\langle 1 \cdot q_{xx} + 1 \cdot \frac{d^2}{dz_1^2} (q) \rangle}_{x_1},$$

which yields

$$\begin{aligned} \llbracket \Delta F, G \rrbracket &= \int dz_1 \int dz_2 \int dx_1 \int dx_2 \int dy_1 \int dy_2 \underbrace{\langle \delta q(z_1), \delta q^\dagger(z_2) \rangle}_{+1} \cdot \\ &\quad \left\langle \left(\underbrace{\frac{d^2}{dy_1^2} (1)}_{x_1} + \underbrace{\frac{d^2}{dz_1^2} (1)}_{x_1} \right) \cdot \underbrace{\frac{d^2}{dy_2^2} (\cos q)}_{x_2} \right\rangle \cdot \underbrace{\langle \delta q(y_1), \delta q^\dagger(y_2) \rangle}_{+1} = 0. \end{aligned}$$

However, from the fact that the other BV-Laplacian,

$$\Delta G = \iiint dz_1 dz_2 dx_2 \underbrace{\langle \delta q(z_1), \delta q^\dagger(z_2) \rangle}_{+1} \cdot \underbrace{\langle 1 \cdot \frac{d^2}{dz_2^2} (-\sin q) \rangle}_{x_2},$$

does not contain q^\dagger so that the first half of the Schouten bracket $\llbracket F, \Delta G \rrbracket$ drops out, we deduce that

$$\begin{aligned} \llbracket F, \Delta G \rrbracket &= \int \cdots \int dz_1 dz_2 dx_1 dx_2 dy_1 dy_2 \underbrace{\langle \delta q(z_1), \delta q^\dagger(z_2) \rangle}_{+1} \cdot \\ &\quad \left\langle \underbrace{(qq_{xx})}_{x_1} \cdot \underbrace{\frac{d^2}{dz_2^2} (-\cos q)}_{x_2} \right\rangle \cdot \underbrace{\langle \delta q^\dagger(y_1), \delta q(y_2) \rangle}_{-1}. \quad (15) \end{aligned}$$

Consequently, the two sides of (11), namely, $\Delta(\llbracket F, G \rrbracket)$ in (14) and $\llbracket \Delta F, G \rrbracket + \llbracket F, \Delta G \rrbracket$ in (15), match perfectly for the functionals F and G at hand.

Proposition 2. *Let $F \in \overline{H}^{n(1+k)}(\pi_{\text{BV}} \times T\pi_{\text{BV}} \times \cdots \times T\pi_{\text{BV}})$ and $G \in \overline{H}^{n(1+\ell)}(\pi_{\text{BV}} \times T\pi_{\text{BV}} \times \cdots \times T\pi_{\text{BV}})$ be two integral functionals; here $k, \ell \geq 0$. Then*

$$\Delta(\llbracket F, G \rrbracket) = \llbracket \Delta F, G \rrbracket + (-)^{\text{gh}(F)-1} \llbracket F, \Delta G \rrbracket. \quad (16)$$

Proof. In the proof we shall not write the evaluations at x_1, x_2, y_1, y_2, z_1 , and z_2 running over the respective copies of M . Likewise, for notational convenience we also do not write the integrations $\int \cdots \int dz_1 dz_2 dx_1 dx_2 dy_1 dy_2$ (see Example 6 above). We have

$$\begin{aligned} \Delta(\llbracket F, G \rrbracket) &= \frac{\overleftarrow{\delta}}{\delta q^\alpha} \frac{\overleftarrow{\delta}}{\delta q_\alpha^\dagger} \left(\frac{\overrightarrow{\delta F}}{\delta q^\beta} \frac{\overleftarrow{\delta G}}{\delta q_\beta^\dagger} - \frac{\overrightarrow{\delta F}}{\delta q_\alpha^\dagger} \frac{\overleftarrow{\delta G}}{\delta q^\beta} \right) \\ &= \frac{\overleftarrow{\delta}}{\delta q^\alpha} \left(\frac{\overleftarrow{\delta}}{\delta q_\alpha^\dagger} \frac{\overrightarrow{\delta F}}{\delta q^\beta} \frac{\overleftarrow{\delta G}}{\delta q_\beta^\dagger} + (-)^{\text{gh}(F)} \frac{\overrightarrow{\delta F}}{\delta q^\beta} \frac{\overleftarrow{\delta}}{\delta q_\alpha^\dagger} \frac{\overleftarrow{\delta G}}{\delta q_\beta^\dagger} - \frac{\overleftarrow{\delta}}{\delta q_\alpha^\dagger} \frac{\overrightarrow{\delta F}}{\delta q_\beta^\dagger} \frac{\overleftarrow{\delta G}}{\delta q^\beta} - (-)^{\text{gh}(F)-1} \frac{\overrightarrow{\delta F}}{\delta q_\beta^\dagger} \frac{\overleftarrow{\delta}}{\delta q_\alpha^\dagger} \frac{\overleftarrow{\delta G}}{\delta q^\beta} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{\overleftarrow{\delta} \overrightarrow{\delta} \overrightarrow{\delta} F}{\delta q^\alpha \delta q_\alpha^\dagger \delta q^\beta \delta q_\beta^\dagger} \frac{\overleftarrow{\delta} \overrightarrow{\delta} G}{\delta q_\alpha^\dagger \delta q_\beta^\dagger} + \frac{\overleftarrow{\delta} \overrightarrow{\delta} F}{\delta q_\alpha^\dagger \delta q_\beta^\dagger} \frac{\overleftarrow{\delta} \overrightarrow{\delta} G}{\delta q^\alpha \delta q^\beta} \\
&\quad + (-)^{\text{gh}(F)} \frac{\overleftarrow{\delta} \overrightarrow{\delta} F}{\delta q^\alpha \delta q^\beta} \frac{\overleftarrow{\delta} \overrightarrow{\delta} G}{\delta q_\alpha^\dagger \delta q_\beta^\dagger} + (-)^{\text{gh}(F)} \frac{\overrightarrow{\delta} \overrightarrow{\delta} F}{\delta q^\beta} \frac{\overleftarrow{\delta} \overrightarrow{\delta} \overleftarrow{\delta} G}{\delta q^\alpha \delta q_\alpha^\dagger \delta q_\beta^\dagger} \\
&\quad - \frac{\overleftarrow{\delta} \overrightarrow{\delta} \overrightarrow{\delta} F}{\delta q^\alpha \delta q_\alpha^\dagger \delta q_\beta^\dagger} \frac{\overleftarrow{\delta} \overrightarrow{\delta} G}{\delta q^\beta} - \frac{\overleftarrow{\delta} \overrightarrow{\delta} F}{\delta q_\alpha^\dagger \delta q_\beta^\dagger} \frac{\overleftarrow{\delta} \overrightarrow{\delta} G}{\delta q^\alpha \delta q^\beta} \\
&\quad - (-)^{\text{gh}(F)-1} \frac{\overleftarrow{\delta} \overrightarrow{\delta} F}{\delta q^\alpha \delta q_\beta^\dagger} \frac{\overleftarrow{\delta} \overrightarrow{\delta} G}{\delta q_\alpha^\dagger \delta q^\beta} - (-)^{\text{gh}(F)-1} \frac{\overrightarrow{\delta} \overrightarrow{\delta} F}{\delta q_\beta^\dagger} \frac{\overleftarrow{\delta} \overrightarrow{\delta} \overleftarrow{\delta} G}{\delta q^\alpha \delta q_\alpha^\dagger \delta q^\beta}.
\end{aligned}$$

As a direct consequence of our approach, the functional derivatives commute – or anticommute if they are both taken with respect to odd-parity coordinates. Applying this observation on the first and fifth terms of the calculation above, we immediately form $\llbracket \Delta F, G \rrbracket$. Similarly, the fourth term (in which we have to swap δq_α^\dagger and δq_β^\dagger in the denominator, yielding an extra sign) and last term combine into $(-)^{\text{gh}(F)-1} \llbracket F, \Delta G \rrbracket$. Thus we obtain

$$\begin{aligned}
\Delta(\llbracket F, G \rrbracket) &= \frac{\overleftarrow{\delta} \overrightarrow{\delta} F}{\delta q_\alpha^\dagger \delta q^\beta} \frac{\overleftarrow{\delta} \overrightarrow{\delta} G}{\delta q^\alpha \delta q_\beta^\dagger} + (-)^{\text{gh}(F)} \frac{\overleftarrow{\delta} \overrightarrow{\delta} F}{\delta q^\alpha \delta q^\beta} \frac{\overleftarrow{\delta} \overrightarrow{\delta} G}{\delta q_\alpha^\dagger \delta q_\beta^\dagger} - \frac{\overleftarrow{\delta} \overrightarrow{\delta} F}{\delta q_\alpha^\dagger \delta q_\beta^\dagger} \frac{\overleftarrow{\delta} \overrightarrow{\delta} G}{\delta q^\alpha \delta q^\beta} \\
&\quad - (-)^{\text{gh}(F)-1} \frac{\overleftarrow{\delta} \overrightarrow{\delta} F}{\delta q^\alpha \delta q_\beta^\dagger} \frac{\overleftarrow{\delta} \overrightarrow{\delta} G}{\delta q_\alpha^\dagger \delta q^\beta} + \llbracket \Delta F, G \rrbracket + (-)^{\text{gh}(F)-1} \llbracket F, \Delta G \rrbracket.
\end{aligned} \tag{17}$$

In the second term, the left factor is symmetric under an exchange of α and β , while the right factor is antisymmetric; therefore it is zero. The third term term vanishes by a similar reasoning. As to the first and fourth terms, if we set the direction of the arrows such that they all point towards the left, then the first term remains the same while the last term picks up an extra sign $(-)^{\text{gh}(F)-1}$, after which it cancels against the first one. Thus, all four terms vanish and the claim follows. \blacksquare

Theorem 3. *Let $F, G \in \overline{\mathfrak{N}}^n(\pi_{\text{BV}})$ be two functionals. The Batalin–Vilkovisky Laplacian Δ satisfies*

$$\Delta(\llbracket F, G \rrbracket) = \llbracket \Delta F, G \rrbracket + (-)^{\text{gh}(F)-1} \llbracket F, \Delta G \rrbracket.$$

In other words, the operator Δ is a graded derivation of the variational Schouten bracket $\llbracket \cdot, \cdot \rrbracket$.

Proof. We prove this by induction over the number of building blocks in each argument of the Schouten bracket in the left hand side of (16). If F and G both belong to $\overline{H}^*(\pi_{\text{BV}} \times T\pi_{\text{BV}} \times \dots T\pi_{\text{BV}})$, then Proposition 2 states the assertion, which is the base of the induction. To make an inductive step, without loss of generality let us assume that the second argument of $\llbracket \cdot, \cdot \rrbracket$ in (16) is a product of two elements from $\overline{\mathfrak{N}}^n(\pi_{\text{BV}})$, each of them containing less multiples from $\overline{H}^*(\pi_{\text{BV}} \times T\pi_{\text{BV}} \times \dots T\pi_{\text{BV}})$ than the product. Denote such factors by G and H and recall that by Theorem 1,

$$\llbracket F, G \cdot H \rrbracket = \llbracket F, G \rrbracket \cdot H + (-)^{(\text{gh}(F)-1) \cdot \text{gh}(G)} G \cdot \llbracket F, H \rrbracket.$$

Therefore, using Theorem 2 we have that

$$\begin{aligned}
&\Delta(\llbracket F, G \cdot H \rrbracket) \\
&= \Delta(\llbracket F, G \rrbracket) \cdot H + (-)^{\text{gh}(F)+\text{gh}(G)-1} \llbracket \llbracket F, G \rrbracket, H \rrbracket + (-)^{\text{gh}(F)+\text{gh}(G)-1} \llbracket F, G \rrbracket \cdot \Delta H \\
&\quad + (-)^{(\text{gh}(F)-1) \text{gh}(G)} \left(\Delta G \cdot \llbracket F, H \rrbracket + (-)^{\text{gh}(G)} \llbracket G, \llbracket F, H \rrbracket \rrbracket + (-)^{\text{gh}(G)} G \cdot \Delta(\llbracket F, H \rrbracket) \right).
\end{aligned}$$

Using the inductive hypothesis in the first and last terms of the right hand side in the above formula, we continue the equality and obtain

$$\begin{aligned}
&= \Delta(\llbracket F, G \rrbracket) \cdot H + (-)^{\text{gh}(F)-1} \llbracket F, \Delta G \rrbracket \cdot H + (-)^{\text{gh}(F)+\text{gh}(G)-1} \llbracket \llbracket F, G \rrbracket, H \rrbracket \\
&\quad + (-)^{\text{gh}(F) \text{ gh}(G)} \llbracket G, \llbracket F, H \rrbracket \rrbracket + (-)^{\text{gh}(F)+\text{gh}(G)-1} \llbracket F, G \rrbracket \cdot \Delta H \\
&\quad + (-)^{(\text{gh}(F)-1) \text{ gh}(G)} \Delta G \cdot \llbracket F, H \rrbracket + (-)^{\text{gh}(F) \text{ gh}(G)} G \cdot \llbracket \Delta F, H \rrbracket \\
&\quad + (-)^{\text{gh}(F) \text{ gh}(G)+\text{gh}(F)-1} G \cdot \llbracket F, \Delta H \rrbracket.
\end{aligned} \tag{18}$$

On the other hand, let us expand the formula

$$\llbracket \Delta F, G \cdot H \rrbracket + (-)^{\text{gh}(F)-1} \llbracket F, \Delta(G \cdot H) \rrbracket,$$

which is the right hand side of (16) in the inductive claim. We obtain

$$\begin{aligned}
&= \llbracket \Delta F, G \rrbracket \cdot H + (-)^{(\text{gh}(\Delta F)-1) \text{ gh}(G)} G \cdot \llbracket \Delta F, H \rrbracket \\
&\quad + (-)^{\text{gh}(F)-1} \llbracket F, \Delta G \cdot H \rrbracket + (-)^{\text{gh}(G)} \llbracket G, H \rrbracket + (-)^{\text{gh}(G)} G \cdot \Delta H \\
&= \llbracket \Delta F, G \rrbracket \cdot H + (-)^{\text{gh}(F) \text{ gh}(G)} G \cdot \llbracket \Delta F, H \rrbracket + (-)^{\text{gh}(F)-1} \llbracket F, \Delta G \rrbracket \cdot H \\
&\quad + (-)^{\text{gh}(F)-1} (-)^{(\text{gh}(F)-1)(\text{gh}(G)-1)} \Delta G \cdot \llbracket F, H \rrbracket + (-)^{\text{gh}(F)-1} (-)^{\text{gh}(G)} \llbracket F, \llbracket G, H \rrbracket \rrbracket \\
&\quad + (-)^{(\text{gh}(F)-1) \text{ gh}(G)} \llbracket F, G \rrbracket \cdot \Delta H \\
&\quad + (-)^{\text{gh}(F)-1} (-)^{\text{gh}(G)} (-)^{(\text{gh}(F)-1) \text{ gh}(G)} G \cdot \llbracket F, \Delta H \rrbracket.
\end{aligned} \tag{19}$$

Comparing (19) with (18), which was derived from the inductive hypothesis, we see that all terms match except for

$$(-)^{\text{gh}(F)+\text{gh}(G)-1} \llbracket \llbracket F, G \rrbracket, H \rrbracket + (-)^{\text{gh}(F) \text{ gh}(G)} \llbracket G, \llbracket F, H \rrbracket \rrbracket$$

from (18) versus

$$(-)^{\text{gh}(F)+\text{gh}(G)-1} \llbracket F, \llbracket G, H \rrbracket \rrbracket$$

from (19). However, these three terms constitute the Jacobi identity (4) for the Schouten bracket. Namely, we have that (c.f. [11])

$$\llbracket F, \llbracket G, H \rrbracket \rrbracket = \llbracket \llbracket F, G \rrbracket, H \rrbracket + (-)^{(\text{gh}(F)-1)(\text{gh}(G)-1)} \llbracket G, \llbracket F, H \rrbracket \rrbracket,$$

so that by multiplying both sides of the identity by $(-)^{\text{gh}(F)+\text{gh}(G)-1}$, we fully balance (18) and (19). This completes the inductive step and concludes the proof. \blacksquare

Lemma 1. *The linear operator*

$$\Delta: \overline{H}^{n(1+k)}(\pi_{\text{BV}} \times T\pi_{\text{BV}} \times \dots \times T\pi_{\text{BV}}) \longrightarrow \overline{H}^{n(2+k)}(\pi_{\text{BV}} \times T\pi_{\text{BV}} \times \dots \times T\pi_{\text{BV}})$$

is a differential for every $k \geq 0$.

Proof. Let $H \in \overline{H}^*(\pi_{\text{BV}} \times T\pi_{\text{BV}} \times \dots \times T\pi_{\text{BV}})$ be an integral functional. Applying the BV-Laplacian Δ twice and again, not writing the integration signs, we obtain

$$\Delta^2(H) = \frac{\delta}{\delta q^\alpha} \frac{\delta}{\delta q_\alpha^\dagger} \frac{\delta}{\delta q^\beta} \frac{\delta}{\delta q_\beta^\dagger} (H) = \frac{\delta}{\delta q^\alpha} \frac{\delta}{\delta q^\beta} \frac{\delta}{\delta q_\alpha^\dagger} \frac{\delta}{\delta q_\beta^\dagger} (H),$$

where we have swapped the middle two functional derivatives in the right hand side. This is the composition of an expression which is symmetric in α and β (the left two functional derivatives) and an expression which is antisymmetric in α and β (the right two functional derivatives). Therefore it is zero. \blacksquare

Theorem 4. *The Batalin–Vilkovisky Laplacian Δ is a differential: for all $H \in \overline{\mathfrak{N}}^n(\pi_{\text{BV}})$ we have*

$$\Delta^2(H) = 0.$$

Proof. We prove Theorem 4 by induction of the number of building blocks from $\overline{H}^*(\pi_{\text{BV}} \times T\pi_{\text{BV}} \times \dots \times T\pi_{\text{BV}})$ in the argument $H \in \overline{\mathfrak{N}}^n(\pi_{\text{BV}})$ of Δ^2 . If $H \in \overline{H}^*(\pi_{\text{BV}} \times T\pi_{\text{BV}} \times \dots \times T\pi_{\text{BV}})$ itself is an integral functional, then by Lemma 1 there remains nothing to prove. Suppose now that $H = F \cdot G$ for some $F, G \in \overline{\mathfrak{N}}^n(\pi_{\text{BV}})$. Then Theorem 2 yields that

$$\Delta^2(F \cdot G) = \Delta \left(\Delta F \cdot G + (-)^{\text{gh}(F)} \llbracket F, G \rrbracket + (-)^{\text{gh}(F)} F \cdot \Delta G \right).$$

Using Theorem 2 again and also Theorem 3, we continue the equality:

$$\begin{aligned} &= \Delta^2 F \cdot G + (-)^{\text{gh}(\Delta F)} \llbracket \Delta F, G \rrbracket + (-)^{\text{gh}(\Delta F)} \Delta F \cdot \Delta G + (-)^{\text{gh}(F)} \llbracket \Delta F, G \rrbracket \\ &\quad + (-)^{\text{gh}(F)} (-)^{\text{gh}(F)-1} \llbracket F, \Delta G \rrbracket + (-)^{\text{gh}(F)} \Delta F \cdot \Delta G \\ &\quad + (-)^{\text{gh}(F)} (-)^{\text{gh}(F)} \llbracket F, \Delta G \rrbracket + (-)^{\text{gh}(F)} (-)^{\text{gh}(F)} F \cdot \Delta^2 G. \end{aligned}$$

By the inductive hypothesis, the first and last terms in the above formula vanish; taking into account that $\text{gh}(\Delta F) = \text{gh}(F) - 1$ in \mathbb{Z}_2 , the terms with $\Delta F \cdot \Delta G$ cancel against each other, as do the terms containing $\llbracket \Delta F, G \rrbracket$ and $\llbracket F, \Delta G \rrbracket$. The proof is complete. \blacksquare

5 The quantum master equation

In this last section we inspect the conditions upon functionals $F \in \overline{\mathfrak{N}}^n(\pi_{\text{BV}})$ under which the Feynman path integrals $\int_{\Gamma(\pi)} [Ds] F([s], [s^\dagger])$ are (infinitesimally) independent of the unphysical, odd-parity antifields $s^\dagger \in \Gamma(\Pi\widehat{\pi})$. The derivation of such a condition (see equation (22) below) relies on an extra assumption of the translation invariance of a measure in the path integral. It must be noted, however, that we do not define Feynman's integral here and do not introduce that measure which essentially depends on the agreement about the classes of ‘admissible’ sections $\Gamma(\pi)$ or $\Gamma(\pi \times_M \Pi\widehat{\pi})$. Consequently, our reasoning is to some extent heuristic.

The basics of path integration, which we recall here for consistency, are standard: they illustrate how the geometry of the BV-Laplacian works in practice. We draw the experts' attention only to the fact that in our notation Ψ is not the gauge fixing fermion Ψ such that $s^\dagger(x) = \Pi(\delta\Psi/\delta q(x))$ but it yields the infinitesimal shift $\dot{q}^\dagger = \delta\Psi/\delta q$ of the anti-objects; we also note that the preservation of parity is not mandatory here and thus an even-parity $\Psi \in \overline{H}^n(\pi) \hookrightarrow \overline{H}^n(\pi_{\text{BV}})$ is a legitimate choice.

Let $F \in \overline{\mathfrak{N}}^n(\pi_{\text{BV}})$ be a functional and $\Psi \in \overline{H}^n(\pi) \hookrightarrow \overline{H}^n(\pi_{\text{BV}})$ be an integral functional which, by assumption, is constant along the antifields: $\Psi(s^\alpha, s_\beta^\dagger) = \Psi(s^\alpha, t_\beta^\dagger)$ for any sections $\{s^\alpha\} \in \Gamma(\pi)$ and $\{s_\beta^\dagger\}, \{t_\beta^\dagger\} \in \Gamma(\Pi\widehat{\pi})$. We investigate under which conditions the path integral $\int_{\Gamma(\pi)} [Ds^\alpha] F(s^\alpha, s_\beta^\dagger) : \Gamma(\Pi\widehat{\pi}) \rightarrow \mathbb{k}$ is infinitesimally independent of a choice of the antifields:

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int_{\Gamma(\pi)} [Ds^\alpha] F \left(s^\alpha, s_\beta^\dagger + \varepsilon \frac{\delta\Psi}{\delta q^\beta(x)} \Big|_{(s^\alpha, s_\beta^\dagger)} \right) = 0 \quad \text{for all } s^\dagger \in \Gamma(\Pi\widehat{\pi}). \quad (20)$$

Note that this formula makes sense because the bundles π and $\widehat{\pi}$ are dual so that a variational covector in the geometry of π acts as a shift vector in the geometry of $\widehat{\pi}$; we let ε be an odd-parity parameter. The left hand side of (20) equals (here and in what follows we proceed over

the building blocks of F by the graded Leibniz rule)

$$\int_{\Gamma(\pi)} [Ds^\alpha] \int_M dx \frac{\overrightarrow{\delta\Psi}}{\delta q^\alpha(x)} \Big|_{(s^\alpha, s_\beta^\dagger)} \cdot \frac{\overleftarrow{\delta F}}{\delta q_\alpha^\dagger(x)} \Big|_{(s^\alpha, s_\beta^\dagger)}, \quad s^\dagger \in \Gamma(\Pi\widehat{\pi}).$$

Take any auxiliary section $\delta s = (\delta s^\alpha, \delta s_\beta^\dagger) \in \Gamma(T(\pi \times_M \Pi\widehat{\pi}))$ normalized by $\delta s^\alpha(x) \cdot \delta s_\alpha^\dagger(x) \equiv 1$ at every $x \in M$ for each $\alpha = 1, \dots, m + m_1 + \dots + m_\lambda = N$ and blow up the scalar integrand to a pointwise contraction of dual object taking their values in the fibres $T_{s(x)}V_x$ and $T_{s^\dagger(x)}\Pi V_x^\dagger$ of $T(\pi \times_M \Pi\widehat{\pi})$ over $x \in M$: for $s = (s^\alpha, s_\beta^\dagger)$ we have

$$\begin{aligned} & \int_M dx \int_M dy \frac{\overrightarrow{\delta\Psi}}{\delta q^\alpha(x)} \Big|_s \cdot \delta_\beta^\alpha \delta(x-y) \cdot \frac{\overleftarrow{\delta F}}{\delta q_\beta^\dagger(y)} \Big|_s \\ &= \int_M dx \int_M dy \frac{\overrightarrow{\delta\Psi}}{\delta q^\alpha(x)} \Big|_s \cdot \delta s^\alpha(x) \langle \vec{e}_\alpha(x), \vec{e}^{\dagger, \beta}(y) \rangle \delta s_\beta^\dagger(y) \cdot \frac{\overleftarrow{\delta F}}{\delta q_\beta^\dagger(y)} \Big|_s. \end{aligned}$$

In fact, the integrand refers to a definition of the evolutionary vector field \mathbf{Q}^Ψ such that $\mathbf{Q}^\Psi(F) \cong \llbracket \Psi, F \rrbracket$ modulo integration by parts in the building blocks of F , c.f. [11]. Due to a special choice of the dependence of Ψ on q only, this is indeed the Schouten bracket $\llbracket \Psi, F \rrbracket$.

To rephrase the indifference of the path integral to a choice of Ψ in terms of an equation upon the functional F alone, we perform integration by parts in Feynman's integral. For this we employ the translation invariance $[Ds] = [D(s - \mu \cdot \delta s)]$ of the functional measure.

Lemma 2. *Let $H \in \overline{\mathfrak{N}}^n(\pi_{\text{BV}})$ be a functional and $\delta s \in \Gamma(T\pi) \hookrightarrow \Gamma(T(\pi \times_M \Pi\widehat{\pi}))$ be a shift section. Then we have that*

$$\int_{\Gamma(\pi)} [Ds^\alpha] \int_M dx \delta s^\alpha(x) \frac{\overleftarrow{\delta H}}{\delta q^\alpha(x)} \Big|_{(s^\alpha, s_\beta^\dagger)} = 0,$$

where the section $s^\dagger \in \Gamma(\Pi\widehat{\pi})$ is a parameter.

Proof. Indeed,

$$0 = \frac{d}{d\mu} \Big|_{\mu=0} \int_{\Gamma(\pi)} [Ds^\alpha] H(s^\alpha, s_\beta^\dagger),$$

because the integral contains no parameter $\mu \in \mathbb{k}$. We continue the equality:

$$\begin{aligned} &= \frac{d}{d\mu} \Big|_{\mu=0} \int_{\Gamma(\pi)} [D(s^\alpha - \mu \delta s^\alpha)] H(s^\alpha, s_\beta^\dagger) \\ &= \frac{d}{d\mu} \Big|_{\mu=0} \int_{\Gamma(\pi)} [Ds^\alpha] H(s^\alpha + \mu \delta s^\alpha, s_\beta^\dagger) = \int_{\Gamma(\pi)} [Ds^\alpha] \frac{d}{d\mu} \Big|_{\mu=0} H(s^\alpha + \mu \delta s^\alpha, s_\beta^\dagger), \end{aligned}$$

which yields the helpful formula in the Lemma's assertion. ■

Returning to the functionals Ψ and F and denoting $G(s) := \frac{d}{d\ell} \Big|_{\ell=0} F(s + \ell \cdot \overleftarrow{\delta s}_\beta^\dagger \cdot \vec{e}^{\dagger, \beta})$, we use the Leibniz rule for the derivative of $H = \Psi \cdot G$:

$$\begin{aligned} \frac{d}{d\mu} \Big|_{\mu=0} (\Psi \cdot G)(s + \mu \cdot \overleftarrow{\delta s}^\alpha \cdot \vec{e}_\alpha) &= \int dx \delta s^\alpha(x) \frac{\overleftarrow{\delta\Psi}}{\delta q^\alpha(x)} \Big|_s \cdot G(s) \\ &\quad + \Psi(s) \cdot \int dx \delta s^\alpha(x) \frac{\overleftarrow{\delta G}}{\delta q^\alpha(x)} \Big|_s. \end{aligned}$$

Because the path integral over $[Ds^\alpha]$ of the entire expression vanishes by Lemma 2, we infer that the path integrals of the two terms are opposite. The integral of the first term equals the initial expression for the path integral over F , i.e., the left-hand side of equation (20). Consequently, if

$$\int_{\Gamma(\pi)} [Ds^\alpha] \Psi(s^\alpha) \cdot \Delta F(s^\alpha, s_\beta^\dagger) = 0 \quad (21)$$

for $\{s_\beta^\dagger\} \in \Gamma(\Pi\widehat{\pi})$ and for all $\Psi \in \overline{H}^n(\pi) \hookrightarrow \overline{H}^n(\pi_{\text{BV}})$, then the path integral over F is infinitesimally independent of the antifields $\{s_\beta^\dagger\} \in \Gamma(\Pi\widehat{\pi})$.

The condition

$$\Delta F = 0 \quad (22)$$

is sufficient for equation (21), and therefore equation (20), to hold. By specifying a class $\Gamma(\pi \times_M \Pi\widehat{\pi})$ of admissible sections of the BV-bundle for a concrete field model, and endowing that space of sections with a suitable metric, one could reinstate a path integral analogue of the main lemma in the calculus of variations and then argue that the condition $\Delta F = 0$ is also necessary.

Summarizing, whenever equation (22) holds, one can assign arbitrary admissible values to the odd-parity antifields; for example, one can let $s_\beta^\dagger(x) = \Pi(\delta\Psi/\delta q^\alpha(x))$ for a gauge-fixing integral $\Psi \in \overline{H}^n(\pi)$. This choice is reminiscent of the substitution principle, see [11] and [19].

Proposition 3. *The \mathbb{k} -linear space of solutions $F \in \overline{\mathfrak{N}}^n(\pi_{\text{BV}})$ to the equation (22) is a graded Poisson algebra with respect to the Schouten bracket $\llbracket \cdot, \cdot \rrbracket$ and the multiplication \cdot in $\overline{\mathfrak{N}}^n(\pi_{\text{BV}})$.*

Proof. The property $\Delta F = 0$ and $\Delta G = 0$ implies $\Delta\llbracket F, G \rrbracket = 0$ by Theorem 3. ■

The Laplace equation (22) ensures the infinitesimal independence from non-physical anti-objects for path integrals of functionals over physical fields – not only in the classical BV-geometry of the bundle $\pi \times_M \Pi\widehat{\pi}$, but also in the quantum setup, whenever all objects are tensored with formal power series $\mathbb{k}[[\hbar, \hbar^{-1}]]$ in the Planck constant \hbar . It is accepted that each quantum field s^\hbar contributes to the expectation value of a functional \mathcal{O}^\hbar with the factor $\exp(iS_{\text{BV}}^\hbar(s^\hbar)/\hbar)$, where S_{BV}^\hbar is the quantum BV-action of the model. Solutions \mathcal{O}^\hbar of the equation $\Delta(\mathcal{O}^\hbar \cdot \exp(iS_{\text{BV}}^\hbar/\hbar)) = 0$ are the *observables*. In particular, the postulate that the unit $1: s \mapsto 1 \in \mathbb{k}$ is averaged to unit by the Feynman integral of $1 \cdot \exp(iS_{\text{BV}}^\hbar(s^\hbar)/\hbar)$ over the space of quantum fields s^\hbar normalizes the integration measure and constrains the quantum BV-action by the quantum master equation (see also, e.g., [2, 3, 9]).

Proposition 4. *Let S_{BV}^\hbar be the even (i.e., it has a density that has an even number of anticommuting coordinates in it) quantum BV-action. If the identity $\Delta(\exp(iS_{\text{BV}}^\hbar/\hbar)) = 0$ holds, then S_{BV}^\hbar satisfies the quantum master equation:*

$$\frac{1}{2}\llbracket S_{\text{BV}}^\hbar, S_{\text{BV}}^\hbar \rrbracket = i\hbar\Delta S_{\text{BV}}^\hbar. \quad (23)$$

We will need the following two lemmas.

Lemma 3. *Let $F \in \overline{H}^n(\pi_{\text{BV}})$ be an even integral functional, let $G \in \overline{\mathfrak{M}}^n(\pi_{\text{BV}})$ be another functional, and let $n \in \mathbb{N}_{\geq 1}$. Then*

$$\llbracket G, F^n \rrbracket = n\llbracket G, F \rrbracket F^{n-1}.$$

Proof. We use induction on Theorem 1. Note that all signs vanish since F is even, meaning that whenever F is multiplied with any other integral functional, the factors may be freely swapped without this resulting in minus signs. For $n = 1$ the statement is trivial. Suppose the formula holds for some $n \in \mathbb{N}_{>1}$, then

$$[[G, F^{n+1}]] = [[G, F \cdot F^n]] = [[G, F]]F^n + F[[G, F^n]] = [[G, F]]F^n + nF[[G, F]]F^{n-1} = (n+1)[[G, F]]F^n,$$

so that the statement also holds for $n + 1$. \blacksquare

Lemma 4. Let $F \in \overline{H}^n(\pi_{\text{BV}})$ be an even integral functional, and let $n \in \mathbb{N}_{\geq 2}$. Then

$$\Delta(F^n) = n(\Delta F)F^{n-1} + \frac{1}{2}n(n-1)[[F, F]]F^{n-2}.$$

Proof. We use induction and the previous lemma. For $n = 2$ the formula clearly holds by Theorem 2. Suppose that it holds for some $n \in \mathbb{N}_{>2}$, then

$$\begin{aligned} \Delta(F^{n+1}) &= \Delta(F \cdot F^n) = (\Delta F)F^n + [[F, F^n]] + F\Delta(F^n) \\ &= (\Delta F)F^n + n[[F, F]]F^{n-1} + nF(\Delta F)F^{n-1} + \frac{1}{2}n(n-1)F[[F, F]]F^{n-2} \\ &= (n+1)(\Delta F)F^n + \frac{1}{2}(n+1)n[[F, F]]F^{n-1}, \end{aligned}$$

so that the statement also holds for $n + 1$. \blacksquare

Proof of Proposition 4. For convenience, we set $F = \frac{i}{\hbar}S_{\text{BV}}^h$. Then

$$\begin{aligned} 0 &= \Delta(\exp F) = \Delta\left(\sum_{n=0}^{\infty} \frac{1}{n!}F^n\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\Delta(F^n) \\ &= \sum_{n=0}^{\infty} \frac{n}{n!}(\Delta F)F^{n-1} + \sum_{n=0}^{\infty} \frac{1}{2n!}n(n-1)[[F, F]]F^{n-2} \\ &= (\Delta F) \sum_{n=1}^{\infty} \frac{1}{(n-1)!}F^{n-1} + \frac{1}{2}[[F, F]] \sum_{n=2}^{\infty} \frac{1}{(n-2)!}F^{n-2} \\ &= \left(\Delta F + \frac{1}{2}[[F, F]]\right) \exp F = \left(\frac{i}{\hbar}\Delta S_{\text{BV}}^h - \frac{1}{2\hbar^2}[[S_{\text{BV}}^h, S_{\text{BV}}^h]]\right) \exp\left(\frac{i}{\hbar}S_{\text{BV}}^h\right), \end{aligned}$$

from which the result follows. \blacksquare

Proposition 5. If the even functionals \mathcal{O} and S_{BV}^h are such that $\Delta(\mathcal{O} \exp(iS_{\text{BV}}^h/\hbar)) = 0$ and $\Delta(\exp(iS_{\text{BV}}^h/\hbar)) = 0$ hold, respectively, then \mathcal{O} satisfies

$$\Omega(\mathcal{O}) := [[S_{\text{BV}}^h, \mathcal{O}]] - i\hbar\Delta\mathcal{O} = 0. \quad (24)$$

Proof. For convenience, let us set $F = \frac{i}{\hbar}S_{\text{BV}}^h$ again. We first calculate, using Lemma 3,

$$[[\mathcal{O}, \exp F]] = \sum_{n=0}^{\infty} \frac{1}{n!}[[\mathcal{O}, F^n]] = \sum_{n=0}^{\infty} \frac{n}{n!}[[\mathcal{O}, F]]F^{n-1} = [[\mathcal{O}, F]] \exp F.$$

Then

$$\begin{aligned} 0 &= \Delta(\mathcal{O} \exp F) = (\Delta\mathcal{O}) \exp F + [[\mathcal{O}, \exp F]] + \mathcal{O}\Delta(\exp F) \\ &= ((\Delta\mathcal{O}) + [[\mathcal{O}, F]]) \exp F = \left((\Delta\mathcal{O}) + \frac{i}{\hbar}[[\mathcal{O}, S_{\text{BV}}^h]]\right) \exp\left(\frac{i}{\hbar}S_{\text{BV}}^h\right), \end{aligned}$$

from which the assertion follows. \blacksquare

Proposition 6. *Let $\mathcal{O} \in \overline{\mathcal{M}}^n(\pi_{\text{BV}})$ be an integral functional, and let the even functional $S_{\text{BV}}^h \in \overline{H}^n(\pi_{\text{BV}})$ satisfy the quantum master equation (23) for the BV-Laplacian Δ . Then the operator Ω , defined in (24), squares to zero:*

$$\Omega^2(\mathcal{O}) = 0.$$

Proof. We calculate, using Theorem 3,

$$\begin{aligned} \Omega^2(\mathcal{O}) &= \llbracket S_{\text{BV}}^h, \llbracket S_{\text{BV}}^h, \mathcal{O} \rrbracket - i\hbar \Delta \mathcal{O} \rrbracket - i\hbar \Delta (\llbracket S_{\text{BV}}^h, \mathcal{O} \rrbracket - i\hbar \Delta \mathcal{O}) \\ &= \llbracket S_{\text{BV}}^h, \llbracket S_{\text{BV}}^h, \mathcal{O} \rrbracket \rrbracket - i\hbar \llbracket S_{\text{BV}}^h, \Delta \mathcal{O} \rrbracket - i\hbar \llbracket \Delta S_{\text{BV}}^h, \mathcal{O} \rrbracket + i\hbar \llbracket S_{\text{BV}}^h, \Delta \mathcal{O} \rrbracket + (i\hbar)^2 \Delta^2 \mathcal{O}. \end{aligned}$$

The last term vanishes identically by Theorem 4, while the second term cancels against the fourth term. Using the Jacobi identity (4) for the Schouten bracket on the first term, we obtain:

$$\Omega^2(\mathcal{O}) = \frac{1}{2} \llbracket \llbracket S_{\text{BV}}^h, S_{\text{BV}}^h \rrbracket, \mathcal{O} \rrbracket - i\hbar \llbracket \Delta S, \mathcal{O} \rrbracket = \frac{1}{2} \llbracket S_{\text{BV}}^h, S_{\text{BV}}^h \rrbracket - i\hbar \Delta S_{\text{BV}}^h, \mathcal{O} \rrbracket = 0,$$

by the quantum master equation for S_{BV}^h . ■

The introduction of the quantum BV-differential Ω concludes our overview of the geometric properties of the Batalin-Vilkovisky Laplacian; yet the story continues with the quantum BV-cohomology theory.

Conclusion

We have outlined a natural class of geometries and described the spaces of admissible functionals (almost none of which are integral functionals except the layer of building blocks such as the action S of the model or the corresponding BV-action S_{BV}) such that the regularization of the BV-Laplacian Δ_{BV} is automatic. The operational definitions of $\llbracket \cdot, \cdot \rrbracket$ and Δ_{BV} and the validity of their intrinsic properties justify the traditional algorithms which are employed in a vast part of the literature to cope with the regularization of δ -functions or infinite constants in this context. At the same time, we expect that the self-regularization mechanism, which we have described and used in this note, can be adapted to and work in a wider set of geometries and formalism that involve double variations or variational derivatives of functionals with respect to pairs of canonically conjugate variables.

We finally remark that the jet-bundle setup and differential calculus do not appeal to commutativity in the models (e.g., to the \mathbb{Z}_2 -graded commutativity if $\pi = (\pi^0 | \pi^1)$ is the superbundle of physical fields). In fact, the entire approach remains valid in the formal noncommutative geometry based on the calculus of cyclic-invariant words (see [11, 13] and references therein).

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